2. Find the particular solution for the following D.Es.

1. $(2x+3)y' = y + \sqrt{2x+3}$, $y(-1) = 0$

2. $y' = x^3 - 2xy$, $y(1) = 1$

Linear Differential Equations of n th order with Constant Coefficients

Linear D.E. of nth order has the form :

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = f(x) \qquad \cdots \qquad (1)$$

or

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x)$$

where $a_1(x), a_2(x), \ldots, a_n(x)$ are functions of x.

- Eq. (1) called D.E. with constant coefficients (معاملات ثابتة) if $a_1, a_2, \ldots, a_n$ constants (ثوابت)
- If $f(x) = 0$, then (1) is called Homogenous
- If $f(x) \neq 0$, then (1) is called non homogenous.

Definition (11): Given $y_1(x), y_2(x), \ldots, y_n(x)$ and the constants $c_1, c_2, \ldots, c_n$ not all zero exist such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = 0 \quad \cdots (2)$$, then $y_1(x)$, $y_2(x), \ldots, y_n(x)$ called linearly dependent functions (معتمدة خطياً), identically in some interval $a \le x \le b$, $y_1(x), y_2(x), \ldots, y_n(x)$ is said to be linearly independent functions (مستقلة خطياً) if eq (2)

implies that $c_1 = c_2 = \cdots = c_n = 0$

مستقلة خطياً

Example (37): show that e^{x}, e^{2x} are linearly independent functions

Solution: From eq(2), we know $c_1 y_1 + c_2 y_2 = 0$, then

$$c_1 e^{x} + c_2 e^{2x} = 0$$

نشتق

$$\mp c_1 e^{x} \mp 2c_2 e^{2x} = 0 \quad \text{بالطرح } 2$$

$$-c_2 e^{2x} = 0$$

Since $e^{2x} \neq 0$, then $c_2 = 0$

$$c_1 e^{x} = 0$$

نعوض في المعادلة الاولى فنجد

Since $e^{2} \neq 0$, then $c_1 = 0$

so $c_1 = c_2 = 0$, then the solution are linearly independent

Example (38): show that the functions $x^2 - x,\ 2x^2,\ 3x$ are linearly dependent

معتمدة خطياً

Solution: Let $c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$, then

$$c_1(x^2 - x) + 2c_2 x^2 + 3c_2 x = 0$$

$$c_1(2x-1) + 4c_2 x + 3c_2 = 0 \quad \text{نشتق}$$

$$2c_1 + 4c_2 = 0 \quad \text{نشتق مرة ثانية}$$

$\Rightarrow c_1 = -2c_2$ نعوضها في المعادلة (2) و(3)

$\Rightarrow 2c_2 x + 3c_2 x = 0,\quad 2c_2 + 3c_3 = 0$

$\Rightarrow c_2 = -\frac{3}{2} c_3,\quad c_1 = 3c_3$

وهذه المعادلات اقل من عدد المجاهيل وبهذه الحالة لايمكن ان يوجد حل وحيد

نفرض ان $c_3 = 2$ نحصل على $c_1 = 6,\ c_2 = -3$

وعليه الدوال $x^2 - x,\ 2x^2,\ 3x$ linearly dependent (معتمدة خطياً)

Theorem (2): If $y_1, y_2, \dots, y_n$ are Solutions of D.E.

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_n y = 0 \quad \text{--- } (3)$$

where $a_1, a_2, \dots, a_n$ constants and $c_1, c_2, \dots, c_n$ also constants then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is solution of homogenous eq. (3)

Theorem (3): we say the function $y(x)$ such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is general solution of D.E. (3), if the solutions $y_1(x), y_2(x), \dots, y_n(x)$ are linearly independent.

Definition (12): The wronskian determinate for the functions $y_1, y_2, \dots, y_n$ defined as

$$w = w(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$$

Theorem (4): If $y_1, y_2, \dots, y_n$ are solutions of D.E. (3) then, these solutions are linearly independent in I iff

$$w(y_1, y_2, \dots, y_n) \neq 0$$

مستقلة خطياً

Example (38): show that e^{x}, e^{2x} linearly independent functions

Solution: from the theorem (4) we note:

$$w(y_1, y_2) = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

then e^{x}, e^{2x} linearly independent functions.

Example (39): show that $\cos x, \sin x$, linearly independent

solution: from wronskian

$$w(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

$\therefore\ y_1 = \cos x,\ y_2 = \sin x$ are linearly independent.

مرتبطة (معتمدة) خطياً

Example (40): show that e^{x}, e^{-x}, e^{2x} linearly dependent in $(-\infty, \infty)$

Solution: from wronskian

$$w(e^{x}, e^{-x}, e^{2x}) = \begin{vmatrix} e^{x} & 2e^{x} & e^{-x} \\ e^{x} & 2e^{x} & -e^{-x} \\ e^{x} & 2e^{x} & e^{-x} \end{vmatrix} = 0$$

نلاحظ فيها صفين متشابهين

then the functions e^{x}, e^{-x}, e^{2x} are linearly dependent functions in $(-\infty, \infty)$.

Example (41): show that x, x^2, x^3 are linearly independent

Solution: Let $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ then

$$w(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \begin{matrix} x & x^2 \\ 1 & 2x \\ 0 & 2 \end{matrix}$$

ضرب المحدد
تكرار العمود الأول
والثاني

$$= (12x^3 + 0 + 2x^3) - (0 + 6x^3 + 6x^3)$$

$$= 14x^3 - 12x^3 = 2x^3 \neq 0$$

$\therefore$ the solutions y_1, y_2, y_3 are linearly independent.

Example (42): prove that e^x, e^{2x} are linearly independent solutions of the D.E. $y'' - 3y' + 2y = 0$.

Solution: Let $y_1 = e^x$, $y_2 = e^{2x}$

first to prove y_1, y_2 are solutions of D.E. we find

$$y_1 = e^x \quad , y_1' = e^x \quad , y_1'' = e^x$$

substituting in D.E.

$$e^x - 3e^x + 2e^x = 0 \Rightarrow 0 = 0 \quad R.H.S = L.H.S$$

$\therefore y_1 = e^x$ is the solution of D.E.

also $y_2 = e^{2x}$, $y_2' = 2e^{2x}$, $y_2'' = 4e^{2x}$

substituting in D.E.

$$4e^{2x} - 3(2e^{2x}) + 2(e^{2x}) = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$$

R.H.S = L.H.S

$\therefore y_2 = e^{2x}$ is solution of D.E.

to prove y_1, y_2 are linearly independent solution, we use wronskian:

$$w(e^{x}, e^{2x}) = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

$\therefore y_1, y_2$ linearly independent solution.

<u>Example (43):</u> Find the values of A and B such that $y = Ax + B$ is the particular solution of D.E. $y'' - 3y' + 2y = 4x$

<u>Solution:</u> $y = Ax + B$, $y' = A$, $y'' = 0$

substituting in D.E.

$$0 - 3A + 2(Ax + B) = 4x$$

$$-3A + 2Ax + 2B$$

$$2Ax + (-3A + 2B) = 4x$$

$$2A = 4 \Rightarrow A = 2$$ نقارن

also $-3A + 2B = 0 \Rightarrow -3(2) + 2B = 0 \Rightarrow B = 3$

Example (44): Find the value of constant A which implies $y = Ax+2$ is solution of D.E. $y'' + 5y' + y = 5x$

Solution: $y = Ax+2$, $y' = A$, $y'' = 0$

substituting in D.E.

$$0 + 5A + (Ax+2) = 5x$$

$$Ax + (5A+2) = 5x \Rightarrow A = 5.$$

Exercises

1. Check the functions for linearly independent or not

a. e^{2x}, e^{3x}, e^{4x} b. $\sin x, \cos x, 1$ c. $e^{x}\sin x, e^{x}\cos x$

2. Prove that $e^{ax}\sin bx$, $e^{ax}\cos bx$ are two solutions of the D.E. $y'' - 2ay' + (a^2+b^2)y = 0$, then prove that these functions are linearly independent.

3. Let w be wronskian for e^{ax}, e^{bx}, e^{cx}, prove

a. $W(x) = W(0)\, e^{(a+b+c)x}$

b. $w(0) = (a-b)(b-c)(c-a)$