Example (25): Find the solution of the D.S.

$$y' = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} y \quad \text{---} (1)$$

ملاحظة: من الممكن ان نكتب النظام بالصيغة

$$y_1' = -2y_1 - 3y_2$$
$$y_2' = 3y_1 + 4y_2$$

Solution: the characteristic eq. of D.S.

$$P_2(r) = \begin{vmatrix} -2-r & -3 \\ 3 & 4-r \end{vmatrix} = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = r_1 = r_2 = 1$$

القيمة الخاصة مكررة

to find the eigen vector, subsitituting in eq.(5), we get

$$(A - rE)V = (A-E)V = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} -2v_{11} - 3v_{12} = 0 \\ 3v_{11} + 4v_{12} = 0 \end{array} \right\} \Rightarrow v_{12} = -v_{11}$$

Let $v_{11} = 1 \Rightarrow v_{12} = -1$, so

$$Y_1(x) = \begin{bmatrix} e^x \\ -e^x \end{bmatrix} , \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to find the second solution we note from the th. (12)

$$Y_2(x) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x e^x + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^x$$

subsitituting $Y_2(x)$ in eq. (1) we get.

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} xe^{x} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{x} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{x} = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} xe^{x} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{x} \right)$$

$$\Rightarrow \begin{bmatrix} B_1xe^{x} + B_1e^{x} + C_1e^{x} \\ B_2xe^{x} + B_2e^{x} + C_2e^{x} \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} B_1xe^{x} + C_1e^{x} \\ B_2xe^{x} + C_2e^{x} \end{bmatrix}$$

نأخذ الصف الاول

$$\Rightarrow B_1xe^{x} + B_1e^{x} + C_1e^{x} = -2B_1xe^{x} - 2C_1e^{x} - 3B_2xe^{x} - 3C_2e^{x}$$

الصف الثاني

$$B_2xe^{x} + B_2e^{x} + C_2e^{x} = 3B_1xe^{x} + 3C_1e^{x} + 4B_2xe^{x} + 4C_2e^{x}$$

$$\Rightarrow B_1xe^{x} + B_1e^{x} + C_1e^{x} + 2B_1xe^{x} + 2C_1e^{x} + 3B_2xe^{x} + 3C_2e^{x} = 0$$

$$(3B_1 + 3B_2)xe^{x} + (B_1 + 3C_1 + 3C_2)e^{x} = 0$$ المعادلة الاولى

$$B_2xe^{x} + B_2e^{x} + C_2e^{x} - 3B_1xe^{x} - 3C_1e^{x} - 4B_2xe^{x} - 4C_2e^{x} = 0$$

$$(3B_1 + 3B_2)xe^{x} + (-B_2 + 3C_1 + 3C_2)e^{x} = 0$$ المعادلة الثانية

$$\Rightarrow \begin{aligned} (3B_1 + 3B_2)xe^{x} + (B_1 + 3C_1 + 3C_2)e^{x} &= 0 \\ (3B_1 + 3B_2)xe^{x} + (-B_2 + 3C_1 + 3C_2)e^{x} &= 0 \end{aligned} \quad \text{--(2)}$$

حل النظام أعلاه

$$\Rightarrow C_3 = -\frac{1}{3}B_1 - C_1, \quad B_2 = -B_1$$

Since B_1, C_1 are arbitrary constant, take $C_1 = 0$, $B_1 = 3$

we get

$$y_2(x) = \begin{bmatrix} 3 \\ -3 \end{bmatrix} xe^{x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{x}$$

the general solution is

$$y(x) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{x} + c_2 \left(\begin{bmatrix} 3 \\ -3 \end{bmatrix} x e^{x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{x} \right)$$

$$\therefore\ y(x) = \begin{bmatrix} c_1 + 3c_2 x \\ -c_1 - 3c_2 x - 1 \end{bmatrix} e^{x}$$

Example(26): solve the D.S.

$$y' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} y \quad \dots (1)$$

solution; the characteristic eq. for the D.S.

$$P_2(r) = \begin{vmatrix} 2-r & 1 \\ -1 & 4-r \end{vmatrix} = 0$$

$$\Rightarrow (2-r)(4-r) + 1 = 0$$

$$\Rightarrow r^2 + 6r + 9 = 0$$

$$\Rightarrow (r-3)^2 = 0 \ , \quad r = r_1 = r_2 = 3$$ جذور مكررة

to find the eigen vector, subsitituting in eq.(5), we find

$$(A - rE)V = (A - 3E)V = \begin{bmatrix} 2-3 & 1 \\ -1 & 4-3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} -v_{11} + v_{12} = 0 \\ -v_{11} + v_{12} = 0 \end{matrix} \Rightarrow v_{11} = v_{12}$$

Let $v_{12} = 1$, then $v_{11} = 1$

thus $y_1(x) = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

to find the second solution, we use th. (12)

$$y_2(x) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x e^{3x} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{3x}$$

Subsitituting $y_2(x)$ in eq. (1), we get

$$\begin{bmatrix} 3B_1 \\ 3B_2 \end{bmatrix} x e^{3x} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{3x} + \begin{bmatrix} 3C_1 \\ 3C_2 \end{bmatrix} e^{3x} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \left(\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x e^{3x} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} e^{3x} \right)$$

$$\Rightarrow \begin{bmatrix} 3B_1 x e^{3x} + B_1 e^{3x} + 3C_1 e^{3x} \\ 3B_2 x e^{3x} + B_2 e^{3x} + 3C_2 e^{3x} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} B_1 x e^{3x} + C_1 e^{3x} \\ B_2 x e^{3x} + C_2 e^{3x} \end{bmatrix}$$

نأخذ الصف الأول

$$3B_1 x e^{3x} + B_1 e^{3x} + 3C_1 e^{3x} = 2B_1 x e^{3x} + 2C_1 e^{3x} + B_2 x e^{3x} + C_2 e^{3x}$$

الصف الثاني

$$\Rightarrow 3B_2 x e^{3x} + B_2 e^{3x} + 3C_2 e^{3x} = -B_1 x e^{3x} - C_1 e^{3x} + 4B_2 x e^{3x} + 4C_2 e^{3x}$$

$$\Rightarrow 3B_1 x e^{3x} + B_1 e^{3x} + 3C_1 e^{3x} - 2B_1 x e^{3x} - 2C_1 e^{3x} - B_2 x e^{3x} - C_2 e^{3x} = 0$$

$$(3B_1 - 2B_1 - B_2) x e^{3x} + (B_1 + 3C_1 - 2C_1 - C_2) e^{3x} = 0$$ المعادلة الأولى

$$3B_2 x e^{3x} + B_2 e^{3x} + 3C_2 e^{3x} + B_1 x e^{3x} + C_1 e^{3x} - 4B_2 x e^{3x} - 4C_2 e^{3x} = 0$$

$$(3B_2 + B_1 - 4B_2) x e^{3x} + (B_2 - C_2 + C_1) e^{3x} = 0$$ المعادلة الثانية

وعليه نحصل على المعادلتين

$$\Rightarrow (B_1 - B_2) x e^{3x} + (B_1 + C_1 - C_2) e^{3x} = 0 \quad \text{--- (2)}$$

$(B_1 - B_2) x e^{3x} + (B_{\cdot}$ [illegible]

وحل النظام (بطريقة الحذف)

$$(B_1-B_2)xe^{3x}+(B_1+C_1-C_2)e^{3x}=0$$

$$\mp(B_1-B_2)xe^{3x}\mp(B_2-C_2+C_1)e^{3x}=0$$

بالطرح

$$(B_1+C_1-C_2-B_2+C_2-C_1)e^{3x}=0$$

$$\Rightarrow (B_1-B_2)e^{3x}=0$$

Since $e^{3x}\neq 0$, then $B_1=B_2$

نعوض في المعادلة الأولى

$$(B_1-B_1)xe^{3x}+(B_1+C_1-C_2)e^{3x}=0$$

$$\Rightarrow (B_1+C_1-C_2)e^{3x}=0$$

Since $e^{3x}\neq 0$, then $C_2=B_1+C_1$

Since B_1, C_1 are arbitrary constant, so let $B_1=C_1=1$

$$\Rightarrow \quad = \quad \Rightarrow B_2=1$$

$$\therefore\ y_2(x)=\begin{bmatrix}1\\1\end{bmatrix}xe^{3x}+\begin{bmatrix}1\\2\end{bmatrix}e^{3x}$$

the general solution

$$y(x)=C_1\begin{bmatrix}e^{3x}\\e^{3x}\end{bmatrix}+C_2\left(\begin{bmatrix}xe^{3x}\\xe^{3x}\end{bmatrix}+\begin{bmatrix}e^{3x}\\2e^{3x}\end{bmatrix}\right)$$

$$y(x)=\begin{bmatrix}C_1+C_2x+1\\C_1+C_2x+2\end{bmatrix}e^{3x}$$

Exercise 2: Solve the following D.Ss

1) $y_1'=y_2$, $y_2'=4y_1$

2) $y'=\begin{bmatrix}3&2\\-5&1\end{bmatrix}y$

3) $y'=\begin{bmatrix}1&-1\\4&5\end{bmatrix}y$