Stability — الاستقرارية

Definition (8): Consider the D.S.

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t) \quad , \quad i = 1, \ldots, n \quad \cdots (1)$$

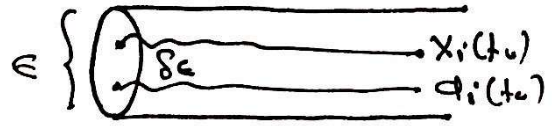
A solution $\phi_i(t)$, $i = 1, 2, \ldots, n$ of D.S. (1) satisfying the initial conditions $\phi_i(t_0) = \phi_{i0}$, $i = 1, 2, \ldots, n$ is said to be stable solution (حل مستقر) as $t \to \infty$ if for any $\epsilon > 0$ $\exists\, \delta_\epsilon > 0$ such that for each solution $x_i(t)$ of D.S. (1) whose initial values satisfy the condition

$$|x_i(t_0) - \phi_i(t_0)| < \delta \quad , \quad i = 1, 2, \ldots, n \quad \cdots (2)$$

the inequalities — t_0 هي اللحظة الابتدائية

$$|x_i(t) - \phi_i(t)| < \epsilon \quad , \quad i = 1, 2, \ldots, n \quad \cdots (3)$$

تحقق hold for all $t \geq t_0$



Definition (9): If under the conditions of eqs (2), (3) the condition $\lim_{t \to \infty} |x_i(t) - \phi_i(t)| = 0$, $i = 1, 2, \ldots, n$ also hold, then the solution $\phi_i(t)$ $i = 1, 2, \ldots, n$ is said to be asymptotically stable (استقرار محاذي)

وعليه سيكون لدينا الحالات الثلاثة

1. stable حل مستقر
2. asymplotically stable حل مستقر محاذي
3. unstable حل غير مستقر

linear system with constant coefficients.

Consider a D.S. of two homogenous linear equations with constant coefficients:

$$\left.\begin{aligned} \frac{dx}{dt} &= ax+by \\ \frac{dy}{dt} &= cx+dy \end{aligned}\right\} \quad \text{--- } (1)$$

النقاط الحرجة

Definition (10): A critical points are the points which implies $x'=0$, $y'=0$ of (1)

for exaple critical points of D.S.

$x' = x+y^2$

$y' = y+x^2$ are the points $(0,0)$, $(-1,-1)$

In order to study the stability of critical point $(0,0)$, we find the roots λ_1, λ_2 of charatersitic eq. and the following cases are possible

I. If λ_1, λ_2 real and distinct حقيقية ومختلفة

In this case the solution of D.S. has the form

$$x(t) = c_1 e^{\lambda_1 t} \quad , \quad y(t) = c_2 e^{\lambda_2 t}$$

we note :

1. if $\lambda_1, \lambda_2 < 0$, then

$$x(t) = c_1 e^{\lambda_1 t} = c_1 e^{-\infty} = 0 \quad \text{as } t \to \infty$$

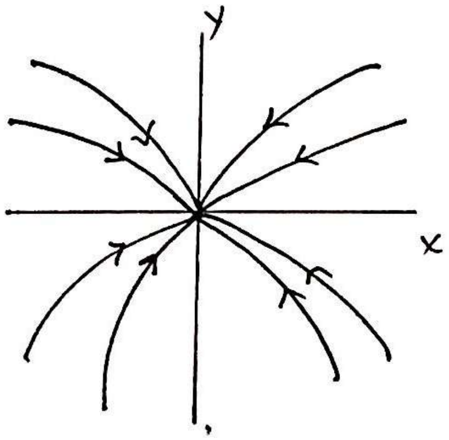
$$y(t) = c_2 e^{\lambda_2 t} = c_2 e^{-\infty} = 0 \quad \text{as } t \to \infty$$

then $x(t) \longrightarrow 0$ as $t \to \infty$

$y(t) \longrightarrow 0$ as $t \to \infty$

مستقرة استقرار تحاذي

$\therefore$ the critical point $(0,0)$ is asymptotical stable and it is called nodal critical point نقطة حرجة عقدية



ملاحظة: اتجاه الاسهم يعني الحل يقترب او يبتعد عن النقطة الحرجة

2. if $\lambda_1, \lambda_2 > 0$ then

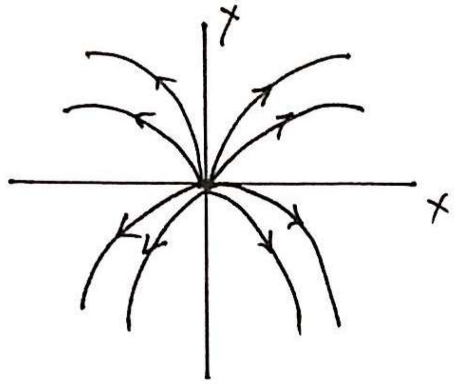
$$x(t) = c_1 e^{\lambda_1 t} = c_1 e^{\infty} = \infty \quad \text{as } t \to \infty$$

$$y(t) = c_2 e^{\lambda_2 t} = c_2 e^{\infty} = \infty \quad \text{as } t \to \infty$$

then $x(t), y(t) \longrightarrow \infty$ as $t \to \infty$

$\therefore$ the critical point is unstable غير مستقرة

and it is called nodal critical point نقطة حرجة عقدية



3. if $\lambda_1 > 0$, $\lambda_2 < 0$ then,

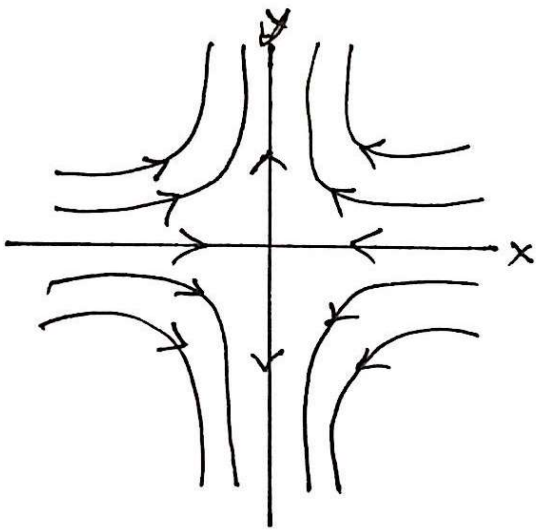
$$x(t) = c_1 e^{\lambda_1 t} = c_1 e^{\infty} = \infty \quad \text{as } t \to \infty$$

$$y(t) = c_2 e^{\lambda_2 t} = c_2 e^{-\infty} = 0 \quad \text{as } t \to \infty$$

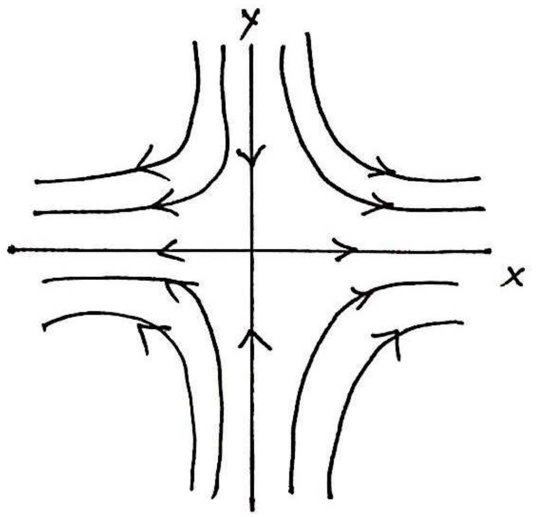
then $x(t) \to \infty$ as $t \to \infty$

$y(t) \to 0$ as $t \to \infty$

then the critical point $(0,0)$ is ustable (غير مستقرة) and it is called saddle critical point نقطة حرجة سرجية



$\lambda_1 < 0, \lambda_2 > 0$



$\lambda_1 > 0$, $\lambda_2 < 0$

II. If λ_1, λ_2 complex i.e. $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$

In this case the solution of D.S. has the form:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$= c_1 e^{(\alpha + i\beta)t} + c_2 e^{(\alpha - i\beta)t}$$
$$= c_1 e^{\alpha t} \cdot e^{i\beta t} + c_2 e^{\alpha t} \cdot e^{-i\beta t}$$
$$= e^{\alpha t}\left[c_1(\cos\beta t + i \sin\beta t) + c_2(\cos\beta t - i \sin\beta t)\right]$$
$$= e^{\alpha t}\left[(c_1 + c_2)\cos\beta t + i(c_1 - c_2)\sin\beta t\right]$$

$$x(t) = e^{\alpha t}\left[c_3 \cos\beta t + c_4 \sin\beta t\right]$$

وكذلك بنفس الاسلوب نجد $y(t)$

we note:

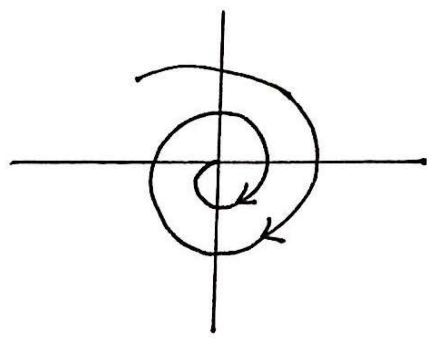
1. if $\alpha < 0$ then

$$x(t) = e^{-\infty}\left[c_3 \cos\beta t + c_4 \sin\beta t\right] \text{ as } t \to \infty$$

$$x(t) = 0$$

the critical point $(0,0)$ asymptotical stable and it is called spiral critical point or (focus)

نقطة حرجة حلزونية   بؤرة



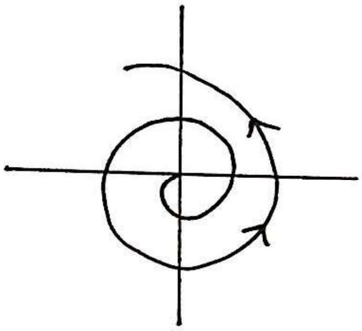
2. if $\alpha > 0$, then

$$x(t) = e^{\infty}\left[C_3 \cos \beta t + C_4 \sin \beta t\right] \text{ as } t \to \infty$$

$$x(t) = \infty$$

حلزونية

the critical point $(0,0)$ unstable and it is called spiral (focus) critical point



3. if $\alpha = 0$, then, $e^{0} = 1$

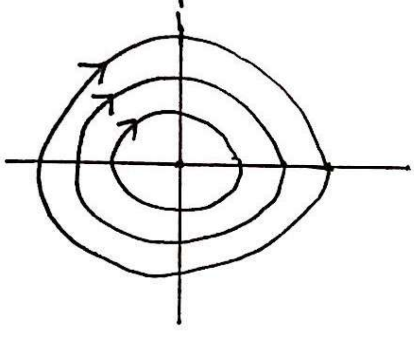
$$x(t) = C_3 \cos \beta t + C_4 \sin \beta t$$

مستقرة او غير مستقرة (مشوش)

the Critical point $(0,0)$ may be stable or unstable, it

نقطة حرجة مركزية

is called Center Critical point



III. if λ_1, λ_2 repeated i.e. $\lambda = \lambda_1 = \lambda_2$

In this case, we note:

if $b=c=0$ in D.S.(1), then D.S.(1) has the form

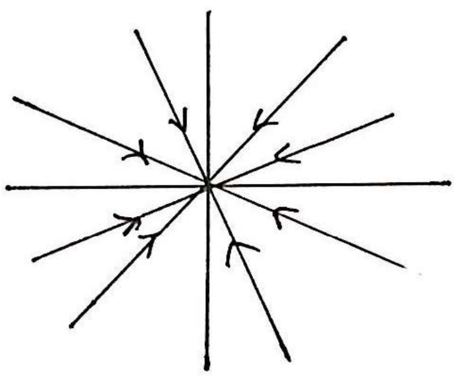
$$x' = ax$$
$$y' = dy$$

and it has solution of the form:

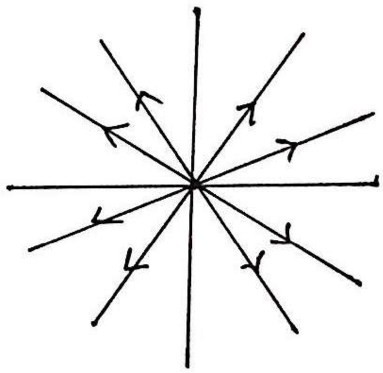
$$x(t) = c_1 e^{\lambda t}, \quad y(t) = c_2 e^{\lambda t}$$

عقدة فعلية

the critical point $(0,0)$ is called proper nodal and it is asymptotical stable if $a+d<0$, and unstable if $a+d>0$



$a+d<0$



$a+b>0$

Example (27): Discuss the stability of $(0,0)$ of the following D.S. and sketch.

1. $x' = -3x + y$
   $y' = 2x - 2y$

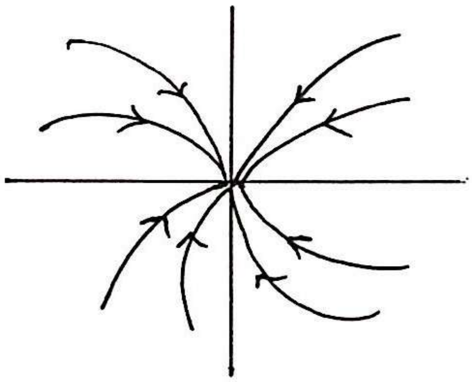
solution: the characteristic eq. المعادلة المميزة

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(-2-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0 \Rightarrow (\lambda+1)(\lambda+4) = 0$$

$$\Rightarrow \lambda_1 = -1 \ , \lambda_2 = -4$$ جذور حقيقية مختلفة

Since $\lambda_1, \lambda_2 < 0$, then the critical point $(0,0)$ is asymphotic stable and it is called nodal (node) عقدية



2. $x' = 2x + y$
   $y' = x + 2y$

Solution: the characteristic eq.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda-3)(\lambda-1) = 0$$

$$\Rightarrow \lambda_1 = 3 \ , \lambda_2 = 1$$ جذور حقيقية مختلفة

Since $\lambda_1, \lambda_2 > 0$, then the critical point $(0,0)$ is unstable غير مستقرة