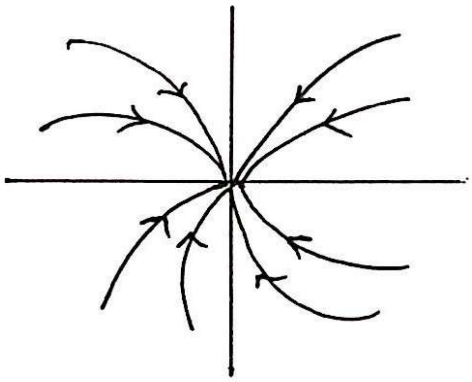
$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-3-\lambda)(-2-\lambda)-2=0$$

$$\Rightarrow \lambda^2+5\lambda+4=0 \Rightarrow (\lambda+1)(\lambda+4)=0$$

$$\Rightarrow \lambda_1=-1 \ , \ \lambda_2=-4$$ جذور حقيقية مختلفة

Since $\lambda_1, \lambda_2 < 0$, then the critical point $(0,0)$ is a symphotic stable and it is called nodal (node) عقدية



2. $x' = 2x + y$
$y' = x + 2y$

Solution: the characteristic eq.

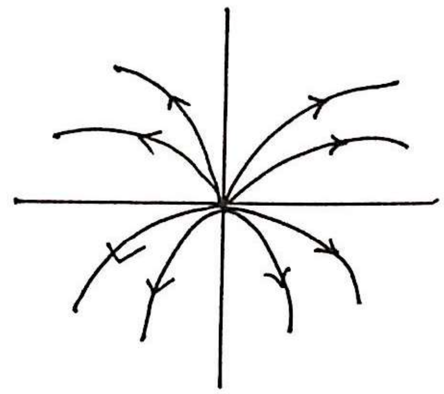
$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(2-\lambda)-1=0$$

$$\Rightarrow \lambda^2-4\lambda+3=0 \Rightarrow (\lambda-3)(\lambda-1)=0$$

$$\Rightarrow \lambda_1=3 \ , \ \lambda_2=1$$ جذور حقيقية مختلفة

Since $\lambda_1, \lambda_2 > 0$, then the critical point $(0,0)$ is unstable غير مستقرة

and it is called nodal عقدية



3. $x' = -x$

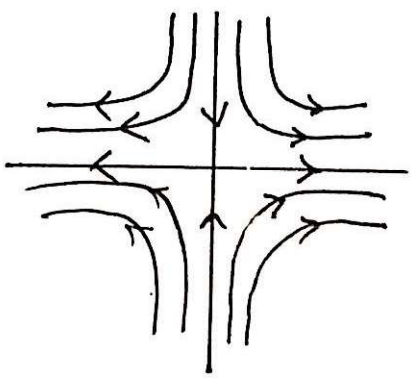
$y' = y$

Solution: the characteristic eq. of the system is

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \mp 1 \Rightarrow \lambda_1 = 1 , \lambda_2 = -1$$

Since $\lambda_1 > 0$, $\lambda_2 < 0$, the critical point $(0,0)$ unstable غير مستقرة
and it is called saddle critical point. نقطة حرجة سرجية



Example (28): Discuss the stability of the critical point (0,0) of the following D.S. and sketch.

1. $x' = y$

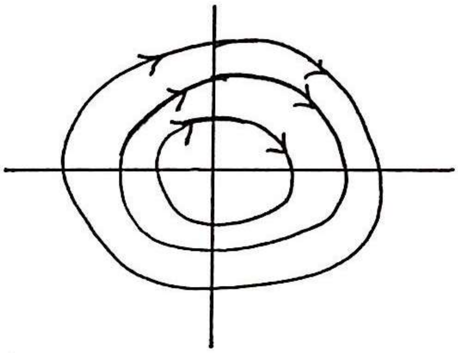
$y' = -x$

Solution: the characteristic eq.

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \mp i$$

$\therefore \lambda_1 = i \ , \ \lambda_2 = -i$ جذور عقدية

thus the critical point (0,0) is called center مركزية and it is stable or unstable



2. $x' = x - 2y$

$y' = 3x - 3y$

Solution: the characteristic eq. is

$$\begin{vmatrix} 1-\lambda & -2 \\ 3 & -3 \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-3-\lambda) + 6 = 0$$

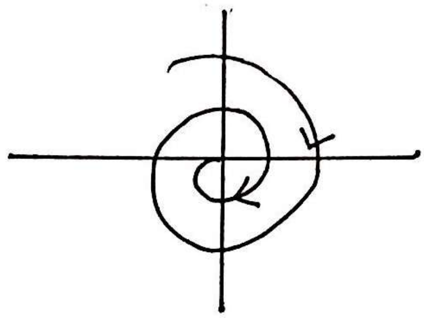
$$\lambda^2 + 2\lambda + 3 = 0$$

$$\lambda = \frac{-2 \mp \sqrt{4 - 4.(1)(3)}}{2(1)} = \frac{-2 \mp 2\sqrt{2}i}{2}$$

$\lambda_1 = -1 + \sqrt{2}\, i \quad , \lambda_2 = -1 - \sqrt{2}\, i$ جذور عقدية

والجزء الحقيقي -1

Since $\alpha = -1 < 0$, the the critical point $(0,0)$ is asymptotical stable and it is called spiral حلزونية



3. $x' = 2x + 2y$
$y' = -x + 3y$

Solution: the characteristic eq. is

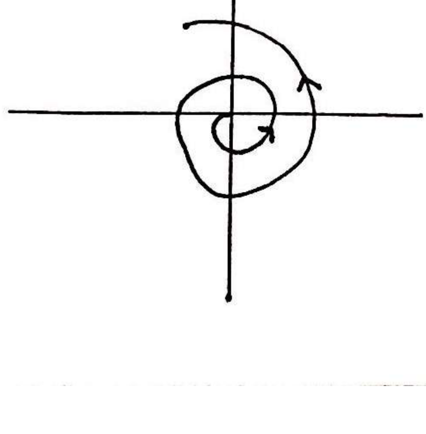
$$\begin{vmatrix} 2-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 8 = 0$$

$$\lambda = \frac{5 \mp \sqrt{25 - 4(1)(8)}}{2(1)} \Rightarrow \lambda = \frac{5 \mp \sqrt{-7}}{2}$$

$\lambda_1 = \frac{5}{2} + \frac{\sqrt{7}}{2} i \quad , \lambda_2 = \frac{5}{2} - \frac{\sqrt{7}}{2} i$ جذور عقدية

Since $\alpha = \frac{5}{2} > 0$, then the critical point $(0,0)$ is unstable and it is called spiral حلزونية



<u>Example (29)</u>: Discuss the stability of $(0,0)$ of the following D.S, and sketch

1. $x' = -2x$
   $y' = -2y$

<u>Solution</u>: the characteristic eq. is

$$\begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)(-2-\lambda) = 0$$

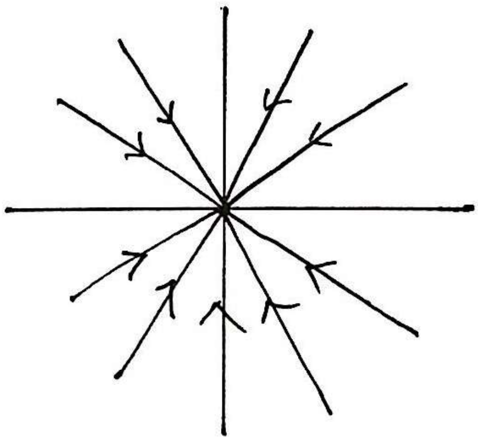
$$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda+2)(\lambda+2) = 0$$

$\Rightarrow \lambda = \lambda_1 = \lambda_2 = 2$ حقيقية مكرر ,δ

the critical point $(0,0)$ is proper nodal عقدية نقطة

and since $a+d = -4 < 0$, then it is asymptotical stable



2. $x' = 8x - y$
   $y' = 4x + 4y$

<u>Solution</u>: the characteristic eq. is

$$\begin{vmatrix} 8-\lambda & -1 \\ 4 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (8-\lambda)(4-\lambda)+4=0$$

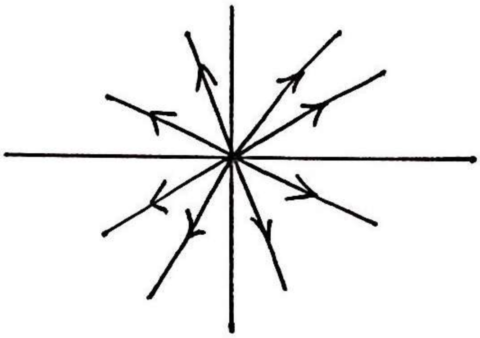
$$\Rightarrow \lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda-6)(\lambda-6)=0$$

$\therefore \lambda = \lambda_1 = \lambda_2 = 6$ الجذور الحقيقية مكررة

$\therefore$ the critical point $(0,0)$ is proper nodal عقدية فعلية

and since $a+d=12>0$, then it is unstable غير مستقرة



The method of Liapunov function طريقة دالة ليابونوف

The method of Liapunov function (Liapunov's direct method)

مباشرة الاستقرارية غير الخطية

is to investigate directly to the stability of nonlinear D.S. the Liapunov function selected as a function $V(x_1, x_2, \dots, x_n)$ has the general form

$$V(x_1, x_2) = a x_1^2 + b x_2^2 \quad , \; a, b \text{ constants}, \; \text{s.t. } V>0$$

$$V(x_1, x_2) \neq (0,0) \quad , \; V(0,0)=0$$

Theorem (13): If for a D.S.

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n) \quad , \; i=1,2,\dots,n \quad \text{--- (1)}$$

there exists a function $V(x_1, x_2, \dots, x_n)$ whose

المشتقة الكلية
Total derivative is:

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \cdot f_i(x_1, \dots, x_n)$$

then we note the following

- If $\frac{dv}{dt} < 0$, then the D.S. has asymptically stable solution مستقر محاذي
- If $\frac{dv}{dt} = 0$, then the D.S. has stable solution حل مستقر
- If $\frac{dv}{dt} > 0$, then the D.S. has unstable solution. غير مستقر

Example (30): Discuss the stability of D.S. by Liapunov method

1. $\dot{x} = y - x^3$

   $\dot{y} = -x - 3y^3$

Solution: we choose the function $V(x,y) = x^2 + y^2$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x(y - x^3) + 2y(-x - 3y^3)$$

$$= -2x^4 - 6y^4 = -2(x^4 + 3y^4) < 0 \quad \forall (x,y) \neq (0,0)$$

$\therefore$ the solution of D.S. is asymptically stable.

2. $\dot{x} = -xy^4$

$\dot{y} = x^4 y$

<u>solution;</u> we choose $V(x,y) = x^4 + y^4$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dt}$$

$$= 4x^3(-xy^4) + 4y^3(x^4y)$$

$$= -4x^4y^4 + 4x^4y^4 = 0$$

$\therefore$ the solution of D.S. is stable

3. $\dot{x} = x + 2xy^2$

$\dot{y} = -2y + 4x^2y$

<u>solution;</u> we choose $V(x,y) = x^2 + y^2$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2x(x + 2xy^2) + 2y(-2y + 4x^2y)$$

$$= 2x^2 + 12x^2y^2 - 4y^2 > 0 \quad \forall (x,y) \neq (0,0)$$

$\therefore$ the solution of D.S. is unstable.

Exercises Discuss the stability of $(0,0)$ for the following D.Es.

1. $x' = 2x$
   $y' = 3y$

2. $x' = -x - 2y$
   $y' = 4x - 5y$

3. $x' = 4x - 2y$
   $y' = 5x + 2y$

4. $x' = 5x + 2y$
   $y' = -17 - 5y$

5. $x' = -3x^3 - y$
   $y' = x^5 - 2y^3$

6. $x' = -2x + xy^3$
   $y' = -x^2y^2 - y^3$

7. $x' = 2xy + x^3$
   $y' = -x^2 + y^5$