Example: Solve the D.S.

$$Y' = AY, \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution: the characteristic eq.

$$P_3(r) = \begin{vmatrix} 1-r & 2 & -1 \\ 0 & 1-r & 1 \\ 0 & -1 & 1-r \end{vmatrix} = 0$$

$$\Rightarrow (1-r)(r^2-2r+2) = 0$$

has roots $r_1 = 1$, $r_2 = 1+i$, $r_3 = 1-i$

to find the eigen value for $r_1 = 1$, we solve

$(A - r_1 I)V = 0$

$$\Rightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2v_{12} - v_{13} &= 0 \\ v_{13} &= 0 \\ -v_{12} &= 0 \end{aligned} \quad \Rightarrow v_{13} = v_{12} = 0 \Rightarrow \text{let } v_{11} = 1$$

so the eign vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $r_2 = 1+i$, we solve

$$\begin{bmatrix} -i & 2 & -1 \\ 0 & -i & 1 \\ 0 & -1 & -i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -iv_{21} + 2v_{22} - v_{23} = 0$$

$$-iv_{22} + v_{23} = 0$$

$$-v_{22} - iv_{23} = 0 \Longrightarrow v_{22} = -iv_{23}$$

So Let $v_{23} = -1 \Rightarrow v_{22} = i \Rightarrow v_{21} = 2-i$

so the eigen vector is $\begin{bmatrix} 2-i \\ 0+i \\ -1+0i \end{bmatrix}$

the eigen vector for $r_3 = 1-i$ will be $\begin{bmatrix} 2+i \\ 0-i \\ -1+0i \end{bmatrix}$

thus the general solution

$$y = c_1 v_1 e^{r_1 x} + c_2 v_2 e^{r_2 x} + c_3 e^{r_3 x} v_3$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 2-i \\ 0+i \\ -1+0i \end{bmatrix} e^{(1+i)x} + c_3 \begin{bmatrix} 2+i \\ 0-i \\ -1 \end{bmatrix} e^{(1-i)x}$$

$$y = \begin{bmatrix} c_1 e^{x} + c_2(2-i) e^{(1+i)x} + c_3 (2+i) e^{(1-i)x} \\ 0.e^{x} + c_2(0+i) e^{(1+i)x} + c_3 (0-i) e^{(1-i)x} \\ 0.e^{x} + c_2(-1+0i) e^{(1+i)x} + c_3(-1) e^{(1-i)x} \end{bmatrix}$$

by using Euler's formula and combining real and imaginary parts, we get

$$y = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{x} + e^{x} \left[c_2 \left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \cos x - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \sin x \right\} \right.$$

$$\left. + c_3 \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cos x + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \sin x \right\} \right]$$

Exercises Solve the following

1) $X' = AX$, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

2) $x' = 2x + y + 2z$
$y' = x + 2y + 2z$
$z' = x + y + 3z$

3) $Y' = AY$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

Non-Homogeneous systems

Consider the system

$$X' = AX + B$$

where $\underset{\sim}{A}$ is a constant $n \times n$ matrix and $\underset{\sim}{B}$ is a vector function of t, then the general solution for the sys. is $X = X_p + X_c$

To find X_c, we solve the homogeneous system

$$X' = AX$$

To find X_p, we will use variation of parameters technique, as in the following examples.

Example: Solve the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{t} \\ 1 \end{bmatrix} \quad \cdots (1)$$

Solution: to find X_c, we solve the homo. sys.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \cdots (2)$$

the char. eq. is

$$P_2(r) = \begin{vmatrix} -r & 1 \\ -2 & 3-r \end{vmatrix} = 0 \Rightarrow r^2 - 3r + 2 = 0$$

$$\Rightarrow (r-2)(r-1) = 0$$

$$\therefore r_1 = 1, \; r_2 = 2$$

The eigen vectors for $r_1 = 1$, $r_2 = 2$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

thus $X_c = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + a_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$, where a_1, a_2 are arbitrary constants

now Let $y_p = a_1(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + a_2(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

Direct substitution into the origin system gives

$$a_1(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + 2a_2(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + a_1'(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + a_2'(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_1(t) e^{t} + \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} a_2(t) e^{2t} + \begin{bmatrix} e^{t} \\ 1 \end{bmatrix} \quad \cdots (3)$$

or more simply

$$a_1'(t)\begin{bmatrix}1\\1\end{bmatrix}e^{t}+a_2'(t)\begin{bmatrix}1\\2\end{bmatrix}e^{2t}=\begin{bmatrix}e^{t}\\1\end{bmatrix} \quad \cdots (4)$$

the other term in (3) cancel each other precisely because X_c is a solution of (2), equation (4) can now be written

$$\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}a_1'(t)e^{t}\\a_2'(t)e^{2t}\end{bmatrix}=\begin{bmatrix}e^{t}\\1\end{bmatrix}$$

using Cramer's rule, we find

$$a_1'(t)e^{t}=\frac{\begin{vmatrix}e^{t} & 1\\1 & 2\end{vmatrix}}{\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix}}=2e^{t}-1$$

$$a_2'(t)e^{2t}=\frac{\begin{vmatrix}1 & e^{t}\\1 & 1\end{vmatrix}}{\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix}}=1-e^{t}$$

thus

$$a_1'(t)=(2e^{t}-1)e^{-t}=2-e^{-t}$$

$$a_2'(t)=(1-e^{t})e^{-2t}=e^{-2t}-e^{-t}$$

so that

$$a_1(t)=\int(2-e^{-t})\,dt=2t+e^{-t}$$

$$a_2(t)=\int(e^{-2t}-e^{-t})\,dt=-\frac{1}{2}e^{-2t}+e^{-t}$$

thus

$$X_p=(2t+e^{-t})\begin{bmatrix}1\\1\end{bmatrix}e^{t}+\left(-\frac{1}{2}e^{-2t}+e^{-t}\right)\begin{bmatrix}1\\2\end{bmatrix}e^{2t}.$$

or $$X_p = (2t * e^{t} + 1)\begin{bmatrix}1\\1\end{bmatrix} + (e^{t} - \tfrac{1}{2})\begin{bmatrix}1\\2\end{bmatrix}$$

$\therefore$ the general solution of system (1) is

$$X = a_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} + a_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t} + (2te^{t}+1)\begin{bmatrix}1\\1\end{bmatrix} + (e^{t}-\tfrac{1}{2})\begin{bmatrix}1\\2\end{bmatrix}$$

Example; Solve the system

$$X' = AX + B \text{ for } A = \begin{bmatrix}2 & 1\\-4 & 2\end{bmatrix},\ B = \begin{bmatrix}3e^{2t}\\te^{2t}\end{bmatrix} \quad \text{--- (5)}$$

Solution; to find X_c, we solve the system

$$X' = AX$$

so

$$P_2(r) = \begin{vmatrix}2-r & 1\\-4 & 2-r\end{vmatrix} = 0 \Rightarrow r^2 - 4r + 8 = 0$$

$$r_1 = 2+2i,\ r_2 = 2-2i$$

so the eigen vectors for r_1 and r_2 are

$$v_1 = \begin{bmatrix}1\\2i\end{bmatrix} = \begin{bmatrix}1+0i\\0+2i\end{bmatrix},\quad v_2 = \begin{bmatrix}1-0i\\0-2i\end{bmatrix}$$

$$\therefore X_c = e^{at}\left[b_1\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\2\end{bmatrix}\sin 2t\right\} + b_2\left\{\begin{bmatrix}0\\2\end{bmatrix}\cos 2t + \begin{bmatrix}1\\0\end{bmatrix}\sin 2t\right\}\right. \quad \text{--- (6)}$$

we seek a particular solution of non homogen. (5) of the form

$$X_p = e^{2t}\left[b_1(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\2\end{bmatrix}\sin 2t\right\} + b_2(t)\left\{\begin{bmatrix}0\\2\end{bmatrix}\cos 2t + \begin{bmatrix}1\\0\end{bmatrix}\sin 2t\right\}\right]$$

Omitting the terms that cancel one another, when X_p is substituted into (5), we have

$$e^{2t}\left[b_1'(t)\left\{\begin{bmatrix}1\\0\end{bmatrix}\cos 2t-\begin{bmatrix}0\\2\end{bmatrix}\sin 2t\right\}+b_2'(t)\left\{\begin{bmatrix}0\\2\end{bmatrix}\cos 2t+\begin{bmatrix}1\\0\end{bmatrix}\sin 2t\right\}\right]=\begin{bmatrix}3e^{2t}\\te^{2t}\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{bmatrix}\begin{bmatrix}b_1'(t)\\b_2'(t)\end{bmatrix}=\begin{bmatrix}3\\t\end{bmatrix}$$

Solving for $b_1'(t)$, $b_2'(t)$ yields

$$b_1'(t)=\frac{\begin{vmatrix}3 & \sin 2t\\ t & 2\cos 2t\end{vmatrix}}{\begin{vmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{vmatrix}}=\frac{1}{2}(6\cos 2t-t\sin 2t)$$

$$b_2'(t)=\frac{\begin{vmatrix}\cos 2t & 3\\ -2\sin 2t & t\end{vmatrix}}{\begin{vmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{vmatrix}}=\frac{1}{2}(t\cos 2t+6\sin 2t)$$

Integrating these function gives

$$b_1(t)=\frac{1}{8}(2t\cos 2t+11\sin 2t)$$

$$b_2(t)=\frac{1}{8}(2t\sin 2t-11\cos 2t)$$

$$\Rightarrow \quad X_p=e^{2t}\begin{bmatrix}b_1(t)\cos 2t+b_2(t)\sin 2t\\ -2b_1(t)\sin 2t+2b_2(t)\cos 2t\end{bmatrix}=e^{2t}\begin{bmatrix}\frac{1}{4}t\\ -\frac{11}{4}\end{bmatrix}$$

$$\therefore \quad X_p=\frac{1}{4}e^{2t}\begin{bmatrix}t\\-11\end{bmatrix}$$

$\therefore$ the general solution is $X=X_c+\frac{1}{4}e^{2t}\begin{bmatrix}t\\-11\end{bmatrix}$

Exercises Find the general solution of the following

1. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{t} \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} te^{2t} \\ -e^{2t} \end{bmatrix}$