Example (61): Solve the D.E. by variation of parameters

$$y'' + 4y = \sec 2x$$

solution: First we find complete solution, the characteristic eq. of D.E. is

$$k^2 + 4 = 0 \Rightarrow k^2 = -4 \Rightarrow k = \mp 2i$$ جذور معقدة

$$\therefore \; y_c(x) = e^{0x}\left[c_1 \sin 2x + c_2 \cos 2x\right.$$

$$y_c(x) = c_1 \sin 2x + c_2 \cos 2x$$

لإيجاد الحل الخاص

Let $y_p(x) = v_1 \sin 2x + v_2 \cos 2x$

من الشرط الأول

$$v_1' \sin 2x + v_2' \cos 2x = 0 \quad \text{—(1)}$$

من الشرط الثاني

$$2v_1' \cos 2x - 2v_2' \sin 2x = \sec 2x \quad \text{...(2)}$$

by using Grammer method

$$v_1' = \frac{\begin{vmatrix} 0 & \cos 2x \\ \sec 2x & -2\sin 2x \end{vmatrix}}{\begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}} = \frac{-\cos 2x \cdot \sec 2x}{-2\sin^2 2x - 2\cos^2 2x}$$

$$= \frac{-\cos 2x \cdot \sec 2x}{-2(\sin^2 2x + \cos^2 2x)} = \frac{-\cos 2x \cdot \frac{1}{\cos 2x}}{-2} = \frac{1}{2}$$

نكامل فنجد

$$v_1 = \frac{1}{2}x$$

كذلك

$$V_2' = \frac{\begin{vmatrix} \sin 2x & 0 \\ 2\cos 2x & \sec 2x \end{vmatrix}}{\begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}} = \frac{\sin 2x \cdot \sec 2x}{-2(\sin^2 2x + \cos^2 2x)}$$

$$= \frac{\sin 2x \cdot \frac{1}{\cos 2x}}{-2} = -\frac{1}{2}\tan 2x$$

تكامل

$$V_2 = \frac{1}{4}\ln|\cos 2x|$$

نعوض عن V_1 ، V_2 بالحل الخاص فنجد

$$y_p(x) = \frac{x}{2}\sin 2x + \frac{1}{4}\ln|\cos 2x|\cos 2x$$

$\therefore$ the general solution

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1\sin 2x + C_2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\ln|\cos 2x| \cdot \cos 2x$$

Example(62): Solve the D.E. $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$, $x \neq 0$ by variation of parameters.

Solution: first we find complete solution, the charactersitic eq. of the D.E. is

$$k^2 + 4k + 4 = 0 \Rightarrow (k+2)(k+2) = 0 \Rightarrow k = k_1 = k_2 = -2$$

الجذور متساوية

$$\therefore y_c(x) = (C_1 + C_2 x)e^{-2x}$$

$$\therefore y_c(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

الحل المتمم

لايجاد الحل الخاص ، نفرض

$$y_p(x) = v_1 e^{-2x} + v_2 x e^{-2x}$$

من الشرط الاول

$$v_1' e^{-2x} + v_2' x e^{-2x} = 0 \quad \text{——} (1)$$

من الشرط الثاني

$$-2v_1' e^{-2x} + v_2' [e^{-2x} - 2x e^{-2x}] = \frac{e^{-2x}}{x^2}$$

اي

$$-2e^{-2x} \cdot v_1' + (e^{-2x} - 2x e^{-2x}) v_2' = \frac{e^{-2x}}{x^2} \quad \text{——} (2)$$

by using Grammer rule

$$v_1' = \frac{\begin{vmatrix} 0 & x e^{-2x} \\ \frac{e^{-2x}}{x^2} & (e^{-2x} - 2x e^{-2x}) \end{vmatrix}}{\begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (e^{-2x} - 2x e^{-2x}) \end{vmatrix}} = \frac{-x e^{-2x} \cdot \frac{e^{-2x}}{x^2}}{e^{-4x} - 2x e^{-4x} + 2x e^{-4x}} = \frac{-1}{x}$$

نكامل

$$v_1 = -\ln|x|$$

كذلك

$$v_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{e^{-2x}}{x^2} \end{vmatrix}}{\begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & (e^{-2x} - 2x e^{-2x}) \end{vmatrix}} = \frac{1}{x^2} \xRightarrow{\text{نكامل}} v_2 = -\frac{1}{x}$$

نعوض بالحل الخاص

$$y_p(x) = -\ln|x| \cdot e^{-2x} - \frac{1}{x} \cdot x e^{-2x}$$

$$\therefore \ y(x) = y_c(x) + y_p(x)$$

$$\therefore \ y(x) = c_1 e^{-2x} + c_2 x e^{-2x} - \ln|x| e^{-2x} - e^{-2x}$$

Example (63): Solve the D.E. by variation of parameters $y'' - 2y' + y = e^x . \ln x$

solution: first we find the complete solution, the characteristic eq. of the D.E. is

$$k^2 - 2k + 1 = 0 \Rightarrow (k-1)(k-1) = 0 \quad , \; k = k_1 = k_2 = 1$$

جذور متشابهة

$$\therefore y_c(x) = (c_1 + c_2 x) e^x$$

$$\therefore y_c(x) = c_1 e^x + c_2 x e^x$$

لإيجاد الحل الخاص، نفرض

$$y_p(x) = v_1 e^x + v_2 x e^x$$

$$e^x v_1' + x e^x v_2' = 0 \quad \text{--- } (1)$$

الشرط الأول

$$e^x v_1' + (x e^x + e^x) v_2' = e^x \ln x \quad \text{--- } (2)$$

الشرط الثاني

by using Grammer rule method

$$v_1' = \frac{\begin{vmatrix} 0 & x e^x \\ e^x . \ln x & x e^x + e^x \end{vmatrix}}{\begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix}} = -x \ln x$$

$\int x \ln x \, dx$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{dx}{x} \qquad v = \frac{x^2}{2}$

$\int u\,dv = uv - \int v\,du$

تكامل

$$v_1 = -\frac{x^2}{2} \ln x + \frac{1}{4} x^2$$

كذلك

$$v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x . \ln x \end{vmatrix}}{\begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix}} = \ln x$$

تكامل مجرب

$$u_2 = x\ln x - x$$

$\int \ln x\, dx$ بالتجزئة

$u = \ln x \quad dv = dx$

$du = \frac{dx}{x}, \quad v = x$

$\int u\,dv = uv - \int v\,du$

$$\therefore\ y_p(x) = \left(\frac{-x^2}{2}\ln x + \frac{1}{4}x^2\right)e^x + (x\ln x - x)\,xe^x$$

$$\therefore\ y(x) = y_c(x) + y_p(x)$$

$$= c_1e^x + c_2xe^x + \left(\frac{-x^2}{2}\ln x + \frac{1}{4}x^2\right)e^x + (x\ln x - x)xe^x$$

Example (64): solve $y'' + y = \tan x$

Solution: the characteristic eq. of D.E. is

$k^2 + 1 = 0 \Rightarrow k^2 = -1 \Rightarrow k = \mp i$ الجذور معقدة (مركبة)

$$\therefore\ y_c(x) = c_1\sin x + c_2\cos x$$

لايجاد الحل الخاص، نفرض

$$y_p(x) = u_1\sin x + u_2\cos x$$

الشرط الاول

$$u_1'\sin x + u_2'\cos x = 0 \quad ---(1)$$

الشرط الثاني

$$u_1'\cos x - u_2'\sin x = \tan x \quad ---(2)$$

نحل (1) و(2) باستخدام قاعدة كرامر

$$u_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x\tan x}{-\sin^2 x - \cos^2 x} = \frac{-\cos x\tan x}{-(\sin^2 x + \cos^2 x)}$$

$$= \frac{\cos x \cdot \frac{\sin x}{\cos x}}{1} = \sin x$$

تكامل مخب

$v_1 = \cos x$

كذلك

$$v_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x . \tan x}{-(\sin^2 x + \cos^2 x)} = \frac{\sin x . \frac{\sin x}{\cos x}}{-1}$$

$$= -\sin^2 x \cos^{-1} x$$

تكامل

$$v_2 = -\frac{\sin^3 x}{3}$$

نعوض بالحل الخاص

$$y_p(x) = -\cos x \sin x - \frac{\sin^3 x}{3} \cos x$$

$$\therefore \quad y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 \sin x + c_2 \cos x - \cos x \sin x - \frac{\sin^3 x}{3} \cos x$$

Exercises: Using the variation of parameters to find the solution of the following D-E.

1. $y'' + y = \sec x$
2. $y'' + 2y' + y = x^2 . e^{-x}$
3. $y'' + 2y' + y = 4e^{-x} \ln x$
4. $y'' - y = \sin^2 x$

قوانين مهمة

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$