Linear Differential Equations with variable coefficients

المعادلات التفاضلية الخطية ذات المعاملات المتغيرة

Euler Equation معادلة اويلر

Euler equation is linear D.E., which is of n-th order and has the form:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = h(x) \quad \ldots (1)$$

where $a_0, a_1, \ldots, a_n$ - constant, h continuous function on $(-\infty, \infty)$,

suppose $u = \ln x \quad , \quad x > 0$

$u = \ln(-x) \quad , \quad x < 0$

or $u = \ln |x|$

by using chain rule قاعدة السلسلة, we can transform eq.(1) to linear D.E. of n-th order with constant coefficients

i.e. $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{1}{x} \quad \ldots (2)$$

$$\therefore \; x \frac{dy}{dx} = \frac{dy}{du}$$

from (2) we note

$$\frac{d^2 y}{dx^2} = \frac{1}{x} \frac{d}{dx}\left(\frac{dy}{du}\right) - \frac{1}{x^2} \frac{dy}{du}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{x} \cdot \frac{d^2 y}{du^2} \cdot \frac{du}{dx} - \frac{1}{x^2} \frac{dy}{du} = \frac{1}{x^2} \frac{d^2 y}{du^2} - \frac{1}{x^2} \frac{dy}{du}$$

$$= \frac{1}{x^2}\left(\frac{d^2 y}{du^2} - \frac{dy}{du}\right)$$

$$\therefore\ x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$$

$\vdots$

and so on

when we use the differential operator $D=\frac{d}{du}$ the derivatives become

$$x\frac{dy}{dx}=x\,Dy$$

$$x^2\frac{d^2y}{dx^2}=x^2D^2y=D(D-1)y$$

$$x^3\frac{d^2y}{dx^3}=x^3D^3y=D(D-1)(D-2)y$$

$\vdots$

$$x^n\frac{d^ny}{dx^n}=x^nD^ny=D(D-1)(D-2)\cdots(D-n+1)y$$

now, If $n=2$, $h(x)=0$, the characteristic eq. of ① can be write as,

$P_2(k)=k(k-1)+a_1k+a_2=0$ and if k_1, k_2 are roots of $P_2(k)=0$, then we note

1. If $k_1\neq k_2$ (الجذور حقيقية مختلفة), then the solution of D.E. has the form $y(x)=c_1|x|^{k_1}+c_2|x|^{k_2}$

2. If $k=k_1=k_2$ (الجذور حقيقية متساوية), then the solution has the form $y(x)=(c_1+c_2\ln|x|)\,|x|^k$

3. If $k_1 = a+ib$, $k_2 = a-ib$ (الجذور مركبة), then the solution has the form $y(x) = [c_1 \sin(b \ln|x|) + c_2 \cos(b \ln|x|)]\,|x|^a$

Example (65): Solve the D.E. $xy'' + y' = 0$

Solution: لحل المعادلة التفاضلية بصيغة اويلر نضرب طرفي المعادلة بـ x فتصبح

$x^2y'' + xy' = 0$ is Euler eq. and homogenous

the characteristic eq. is

$$k(k-1) + k = 0$$

$$k^2 - k + k = 0 \Rightarrow k^2 = 0 \Rightarrow k = k_1 = k_2 = 0$$ (الجذور حقيقية متساوية)

$$\therefore\ y(x) = (c_1 + c_2 \ln|x|)\,|x|^0$$

$$= c_1 + c_2 \ln|x|$$

Example (66): Solve the D.E. $x^2y'' + 2xy' - 6y = 0$ with the conditions $y(1) = 1$, $y'(1) = 0$

Solution; the charac. eq.

$$k(k-1) + 2k - 6 = 0$$

$$k^2 - k + 2k - 6 = 0$$

$$k^2 + k - 6 = 0 \Rightarrow (k+3)(k-2) = 0$$

$\therefore\ k_1 = -3$, $k_2 = 2$ الجذور حقيقية مختلفة

$$\therefore\ y(x) = c_1|x|^{-3} + c_2|x|^2$$

$$= c_1 x^{-3} + c_2 x^2$$

$$y'(x) = -3c_1 x^{-4} + 2c_2 x$$

نعوض بالشروط

$1 = c_1 + c_2 \quad \cdots ①$

$0 = -3c_1 + 2c_2 \quad \cdots ②$

بحل المعادلتين نحصل على

$\Rightarrow c_1 = \frac{2}{5}, \quad c_2 = \frac{3}{5}$

$\therefore y(x) = \frac{2}{5}x^{-3} + \frac{3}{5}x^2$ الحل الخاص

Example (67): Solve the D.E. $x^3y''' + 2x^2y'' + xy' - y = 0$

Solution: the characteristic eq. is

$P_3(k) = k(k-1)(k-2) + 2k(k-1) + k - 1 = 0$

$k^3 - k^2 + k - 1 = 0$

$k^2(k-1) + (k-1) = 0$

$(k-1)(k^2+1) = 0 \Rightarrow k_1 = 1, \; k_2 = i, \; k_3 = -i$

$\therefore y(x) = c_1 x + [c_2 \sin(\ln|x|) + c_3 \cos(\ln|x|)]$

Example (68): Solve the D.E.

$$x^2y'' + xy' = x + x^2$$

solution: كون معادلة اويلر غير متجانسة، فنبدأ اولاً بالحل المتمم لها بجعلها متجانسة.

$P_2(k) = k(k-1) + k = 0$ المعادلة المميزة

$k^2 - k + k = 0 \Rightarrow k^2 = 0 \Rightarrow k = k_1 = k_2 = 0$

الجذور متساوية

$\therefore y_c(x) = (c_1 + c_2 \ln|x|)|x|^0$

$= c_1 + c_2 \ln|x|$

لايجاد الحل الخاص y_p نستخدم صيغة المؤثر

Let $t = \ln x \Rightarrow x = e^{t}$

$$y_p(t) = \frac{1}{P_2(D)}\left(e^{t} + e^{2t}\right) = \frac{1}{D^2}\left(e^{t} + e^{2t}\right)$$

$$= \frac{e^{t}}{D^2} + \frac{e^{2t}}{D^2} = \frac{1}{D^2(e^{-t})} + \frac{1}{D^2(e^{-2t})}$$

مشتقة ثانية مشتقة ثانية

$$= \frac{1}{e^{-t}} + \frac{1}{4e^{-2t}}$$

$$\therefore\ y_p(t) = e^{t} + \frac{1}{4}e^{2t} \Rightarrow y_p(x) = x + \frac{1}{4}x^2$$

$$\therefore\ y(x) = y_c(x) + y_p(x) = c_1 + c_2 \ln|x| + x + \frac{1}{4}x^2$$

Example(69): Solve the D.E. $x^2y'' - xy' + y = \ln x$

Solution: لايجاد الحل المتمم نكتب المعادلة المميزة لحالة التجانس

$$K(K-1) - K + 1 = 0$$

$$K^2 - K - K + 1 = 0 \Rightarrow K^2 - 2K + 1 = 0 \Rightarrow (K-1)(K-1) = 0$$

$$\therefore\ K = K_1 = K_2 = 1$$ جذور متساوية

$$\therefore\ y_c(x) = \left[c_1 + c_2 \ln|x|\right]|x|$$

لايجاد الحل الخاص نستخدم طريقة المؤثر

Let $t = \ln x$

$$y_p(t) = \frac{1}{P_2(D)} \cdot t = \frac{1}{D^2 - 2D + 1} \cdot t = \frac{1}{1 + (D^2 + 2D)} \cdot t$$

$$y_p(t) = \left(1 + (D^2 - 2D)\right)^{-1} \cdot t$$

$$= \left(1 - (D^2 - 2D) + (D^2 - 2D)^2 - \cdots\right) \cdot t$$

$$= t + 0 + 2(1)$$

$$= t + 2$$

$$\therefore y_p(x) = \ln x + 2$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= (c_1 + c_2 \ln|x|)|x| + \ln x + 2$$

Let $x = D^2 - 2D$

$$\frac{1}{1 + (D^2 - 2D)} = \frac{1}{1 + x}$$

$\Downarrow$ بالقسمة المطولة

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 + \cdots \\ \hline 1 + x & 1 \\ & \mp 1 \mp x \\ \hline & \pm x \\ & \mp x \pm x^2 \\ \hline & x^2 \\ & \vdots \end{array}$$

بالطرح

$$\therefore \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

Example (70): solve the D.E. $x^2 y'' + x y' + 9y = \sin(\ln x^3)$

Solution: لإيجاد الحل العام لمعادلة اويلر غير المتجانسة ، نحل المعادلة المتجانسة والمعادلة المميزة لها هي

$$P_2(k) = k(k-1) + k + 9 = 0$$

$$\Rightarrow k^2 - k + k + 9 = 0 \Rightarrow k^2 + 9 = 0 \Rightarrow k^2 = -9 \Rightarrow k = \mp 3i$$

جذور مركبة

$$\therefore y_c(x) = c_1 \sin(3\ln|x|) + c_2 \cos(3\ln|x|)$$

لإيجاد الحل الخاص نستخدم صيغة المؤثر

Let $t = \ln x \Rightarrow e^t = x \Rightarrow e^{3t} = x^3 \Rightarrow \ln x^3 = 3t$

$$y_p(t) = \frac{1}{P_2(D)} \sin 3t = \frac{1}{D^2 + 9} \sin 3t = \frac{1}{9\left(1 + \frac{D^2}{9}\right)} \sin 3t$$

$$= \frac{1}{9}\left(1 + \frac{D^2}{9}\right)^{-1} \sin 3t = \frac{1}{9}\left[1 - \frac{D^2}{9} + \frac{D^4}{81} - \cdots\right] \sin 3t$$

$$= \frac{1}{9}\left[\sin 3t + \sin 3t - \sin 3t + \cdots\right] = \frac{1}{9}\sin 3t$$

$$\therefore\ y_p(x) = \frac{1}{9}\sin(\ln x^3)$$

$$y(x) = y_c(x) + y_p(x)$$

$$\therefore\ y(x) = c_1 \sin(3\ln|x|) + c_2\cos(3\ln|x|) + \frac{1}{9}\sin(\ln x^3)$$

ملاحظة:

$$\frac{1}{1-D} = 1 + D + D^2 + D^3 + \cdots$$

$$\frac{1}{1+D} = 1 - D + D^2 - D^3 + \cdots$$

Exercises: Solve the following D.E.

1. $x^2y'' - 2xy' - 10y = 3x + 8$
2. $x^2y'' - xy' + 2y = 3\ln x$
3. $x^2y'' + 4xy' + 2y = e^x$
4. $x^3y''' + 2x^2y' + 2y = 10\left(x + \frac{1}{x}\right)$
5. $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 1 + x^2$
6. $x^3y''' + 2xy' - 2y = x^2\ln x + 3x$.