B) Find the particular solution for the following

1. $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$
2. $y'' + y' - 3y = 0$, $y(0) = 1$, $y'(0) = 1$
3. $9y'' - 3y' - 2y = 0$ $y(0) = 3$, $y'(0) = 1$

Linear non-homogenous D.E. with constant coefficients

There is two methods to find the solution of non-homogenous D.E. :

1. The method of undetermind coefficients (المعاملات غير المحددة (المجهولة
2. Variation of constant (parameters) (تغيير الثوابت (البارمترات

1. The method of undetermind coefficient.

In this method first we find the solution of homogenous D.E. which called complete solution (الحل المتمم) $y_c(x)$ and second we find particular solution (الحل الخاص) $y_p(x)$, the the solution of D.E. is

$$y(x) = y_c(x) + y_p(x).$$

Consider the non homogenous D.E. of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = h(x) \quad \text{——} \quad (1)$$

to find particular solution of (1), this dependent (يعتمد) on the form of h(x) as the following

متعددة حدود درجة

1. If $h(x)$ polynomail of degree n, we suppose the particular solution as:

$$y_p(x) = Ax^n + Bx^{n-1} + \cdots + D \qquad \text{--- (2)}$$

where $A, B, \ldots, D$ constants. we find the values of $A, B, \ldots, D$ and substituting in (2), this implies the particular (الحل الخاص) solution $y_p(x)$

Example (50): Solve the D.E. $y'' + y' - 2y = 2x$

Solution: First we find $y_c(x)$ (الحل المتمم), the characteristic eq. of homogenous D. E. are

$$k^2 + k - 2 = 0 \Rightarrow (k+2)(k-1) = 0 \Rightarrow k_1 = -2, \; k_2 = 1$$

$$\therefore \; y_c(x) = c_1 e^{-2x} + c_2 e^{x}$$

To find the particular solution, we let

$$y_p(x) = Ax + B$$ لايجاد الحل الخاص، نفترض

$$y_p' = A, \quad y_p'' = 0$$

نعوض بالمعادلة الاصلية

$$0 + A - 2(Ax + B) = 2x$$

$$A - 2Ax - 2B = 2x$$

$$(A - 2B) - 2Ax = 2x$$

نعزل

$$\Rightarrow -2A = 2 \Rightarrow A = -1$$

$$A - 2B = 0 \Rightarrow (-1) - 2B = 0 \Rightarrow B = -\frac{1}{2}$$

نعوض

$$y_p(x) = -x - \frac{1}{2} \Rightarrow y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{-2x} + c_2 e^{x} - x - \frac{1}{2}$$

Example (51): solve $y'' - 3y' - 4y = 4x^2$

Solution: First we find $y_c(x)$, the characteristic eq.

$$k^2 - 3k - 4 = 0$$

$$(k-4)(k+1) = 0 \Rightarrow k_1 = 4 \,,\; k_2 = -1$$

$$\therefore\; y_c(x) = c_1 e^{4x} + c_2 e^{-x}$$ الحل المتمم

To find the particular solution $y_p(x)$, suppose

$$y_p(x) = Ax^2 + Bx + C.$$

$$y'_p = 2Ax + B \,,\quad y''_p = 2A$$

نعوض بالمعادلة الاصلية

$$2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2$$

$$2A - 6Ax - 3B - 4Ax^2 - 4Bx - 4C = 4x^2$$

$$(2A - 3B - 4C) + (-6A - 4B)x - 4Ax^2 = 4x^2$$

نقارن

$$-4A = 4 \Rightarrow A = -1$$

$$-6A - 4B = 0 \Rightarrow -6(-1) - 4B = 0 \Rightarrow B = 3/2$$

ندرك

$$2A - 3B - 4C = 0 \Rightarrow 2(-1) - 3\left(\tfrac{3}{2}\right) - 4C = 0 \Rightarrow C = \frac{-13}{8}$$

$$\therefore\; y_p(x) = -x^2 + \frac{3}{2}x - \frac{13}{8}$$ الحل الخاص

$$y(x) = y_c(x) + y_p(x)$$ الحل العام

$$\therefore\; y(x) = c_1 e^{4x} + c_2 e^{-x} - x^2 + \frac{3}{2}x - \frac{13}{8}$$

Example (52): solve the D.E. $y'' + y' - 2y = 2 + 2x - 2x^2$

solution: first we find $y_c(x)$, the characteristic eq.

$$k^2 + k - 2 = 0$$

$$(k+2)(k-1) = 0 \Rightarrow k_1 = -2 \, , \, k_2 = 1$$

$$\therefore y_c(x) = c_1 e^{-2x} + c_2 e^{x} \quad \text{(الحل المتمم)}$$

To find the particular solution $y_p(x)$, Let

$$y_p(x) = Ax^2 + Bx + c$$

$$y_p' = 2Ax + B \quad , \quad y_p'' = 2A$$

نعوض بالمعادلة الاصلية

$$2A + (2Ax + B) - 2(Ax^2 + Bx + c) = 2 + 2x - 2x^2$$

$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2c = 2 + 2x - 2x^2$$

نقارن

$$2A + B - 2C = 2$$

$$2A - 2B = 2$$

$$-2A = -2 \Rightarrow A = 1 \, , \, B = 0 \, , \, C = 0$$

$$\therefore y_p(x) = x^2 \quad \text{(الحل الخاص)}$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{-2x} + c_2 e^{x} + x^2$$

2. If $h(x)$ of the form be^{ax}, where a, b constants, we note the following:

a. If a no-one of the roots of charactersitic eq. then suppose $y_p(x) = Ae^{ax}$ اذا كانت a ليس احد الجذور للمعادلة المميزة

b. If a is one root of the characteristic eq. but not equal to other roots, then suppose $y_p(x) = Axe^{ax}$

اذا كان a احد الجذور للمعادلة المميزة وغير مكرر

c. If a is one roots of the charactersitic equation and equal it n times, suppose $y_p(x) = Ax^n e^{ax}$

اذا كان a احد الجذور ومكرر n من المرات

Example (53): Solve the D.E. $y'' - y' - 2y = 6e^x$

المعادلة المميزة

Solution: First we find $y_c(x)$. the characteristic eq.

$$k^2 - k - 2 = 0 \Rightarrow (k-2)(k+1) = 0 \Rightarrow k_1 = 2 \,,\, k_2 = -1$$

$$\therefore y_c(x) = c_1 e^{2x} + c_2 e^{-x}$$

let $y_p(x) = Ae^x$ كون $a=1$ ليس احد الجذور نفرض

$y_p' = Ae^x \,,\, y_p'' = Ae^x$ نعوض بالمعادلة الاصلية

$$Ae^x - Ae^x - 2Ae^x = 6e^x \Rightarrow -2A = 6 \Rightarrow A = -3$$

نعوض

$\therefore y_p(x) = -3e^x$ الحل الخاص

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^{2x} + c_2 e^{-x} - 3e^x$$

Example (54): Solve the D.E. $y'' - 9y = e^{3x}$

solution: First we find $y_c(x)$, the charactersitic eq.

$k^2 - 9 = 0 \Rightarrow k = \mp 3 \Rightarrow k_1 = 3 \,,\, k_2 = -3$ جذور حقيقية مختلفة

$y_c(x) = c_1 e^{3x} + c_2 e^{-3x}$ الحل المتمم

نلاحظات

نلاحظات احد الجذور هو (3) وهو قيمة $a=3$ في e^{3x} كذلك

Let $y_p(x) = A x e^{3x}$

$$y_p' = A\left[e^{3x} + 3x e^{3x}\right] = A e^{3x} + 3Ax e^{3x}$$

$$y_p'' = 3A e^{3x} + 3A\left[e^{3x} + 3x e^{3x}\right]$$
$$= 3A e^{3x} + 3A e^{3x} + 9Ax e^{3x}$$
$$= 6A e^{3x} + 9Ax e^{3x}$$

نعوض بالمعادلة الاصلية

$$6A e^{3x} + 9Ax e^{3x} - 9(Ax e^{3x}) = e^{3x}$$

نقارن

$$\Rightarrow 6A e^{3x} = e^{3x} \Rightarrow A = \frac{1}{6}$$

$$\therefore y_p(x) = \frac{x}{6} e^{3x}$$ الحل الخاص

$$y(x) = y_c(x) + y_p(x)$$ الحل العام

$$= c_1 e^{3x} + c_2 e^{-3x} + \frac{x}{6} e^{3x}$$

Example (55): Solve the D.E. $y'' - 2y' + y = 3e^{x}$

Solution: First we find $y_c(x)$, the characteristic eq.

$$k^2 - 2k + 1 = 0$$
$$(k-1)(k-1) = 0 \Rightarrow k_1 = k_2 = 1$$

$$\therefore y_c(x) = c_1 e^{x} + c_2 e^{x}$$

نلاحظ ان $a=1$ في e^{x} وهو احد الجذور ومكرر

Let $y_p(x) = A x^2 e^{x}$

$$y'_p = 2Axe^x + Ax^2e^x$$

$$y''_p = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

نعوض بالمعادلة الاصلية

$$2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x - 2\left[2Axe^x + Ax^2e^x\right] + Ax^2e^x = 3e^x$$

$$e^x\left[2A + \cancel{4Ax} + \cancel{Ax^2} - \cancel{4Ax} - \cancel{2Ax^2} + \cancel{Ax^2}\right] = 3e^x$$

نقارن

$$\Rightarrow 2A = 3 \Rightarrow A = 3/2$$

$$\therefore y_p(x) = \frac{3}{2}x^2e^x$$ الحل الخاص

$$y(x) = y_c(x) + y_p(x)$$ الحل العام

$$= c_1e^x + c_2xe^x + \frac{3}{2}x^2e^x$$

3) If $h(x)$ of the form $h(x) = b\sin ax$ or $h(x) = b\cos ax$, where a, b constant, we note the following:

a. If a no-one of the roots of characteristic eq., then suppose $y_p(x) = A\cos ax + B\sin ax$, A, B constants.

b. If a is one root of characteristic eq., we suppose
$$y_p(x) = x\,(A\cos ax + B\sin ax)$$

c. If a is one of the roots of the charasteristic eq. and repated (equal) n-times, we suppose
$$y_p(x) = x^n(A\cos ax + B\sin ax)$$

Example(56): Solve the D.E. $3y'' - 5y' - 2y = 4\sin 5x$

Solution: The charactersitic eq. of the D.E is

$$3k^2 - 5k - 2 = 0$$

$$(k-2)(3k+1) = 0 \Rightarrow k_1 = 2 \quad , k_2 = -\frac{1}{3}$$ جذور حقيقية مختلفة

$$y_c(x) = c_1 e^{2x} + c_2 e^{-1/3 x}$$

لايجاد الحل الخاص نلاحظ ان قيمة $a = 5$ في المعادلة التفاضلية

وهي ليست احد قيم الجذور k_1, k_2 وعليه

we suppose $y_p(x) = A\cos 5x + B\sin 5x$

$$y_p'(x) = -5A\sin 5x + 5B\cos 5x$$

$$y_p''(x) = -25A\cos 5x - 25B\sin 5x$$

نعوض بالمعادلة الاصلية

$$3(-25A\cos 5x - 25B\sin 5x) - 5(-5A\sin 5x + 5B\cos 5x$$
$$-2(A\cos 5x + B\sin 5x) = 4\sin 5x$$

تبسيط المسألة نجد ان

$$-77\cos 5x + 25A\sin 5x - 25B\cos 5x - 77B\sin 5x = 4\sin 5x$$

$$\Rightarrow (-77A - 25B)\cos 5x + (25A - 77B)\sin 5x = 4\sin 5x$$

نقارن

$$-77A - 25B = 0$$
$$25A - 77B = 4$$

بحل المعادلتين نجد

$$A = 0.015 \quad , B = -0.047$$

$$\therefore y_p(x) = 0.015\cos 5x - 0.047\sin 5x$$

$$y(x) = y_c(x) + y_p(x)$$

$$\therefore y(x) = c_1 e^{2x} + c_2 e^{-\frac{1}{3}x} + 0.015\cos 5x - 0.047\sin 5x$$

Example (57): solve the D.E. $y'' - y = \sin x$

Solution; the characteristic eq of the D.E. is

$$k^2 - 1 = 0 \Rightarrow (k-1)(k+1) = 0 \Rightarrow k = \pm 1$$ جذور حقيقية مختلفة

$$\therefore y_c(x) = c_1 e^{x} + c_2 e^{-x}$$

لايجاد الحل الخاص نلاحظ ان قيمة a=i في المعادلة هي احد جذور المعادلة التفاضلية وعليه

we suppose $y_p(x) = x(A\cos x + B\sin x)$

$$y'_p(x) = x(-A\sin x + B\cos x) + (A\cos x + B\sin x)$$

$$y''_p(x) = x(-A\cos x - B\sin x) + (-A\sin x + B\cos x) + (-A\sin x + B\cos x)$$

$$y''_p(x) = x(-A\cos x - B\sin x) - 2A\sin x + 2B\cos x$$

نعوض بالمعادلة الاصلية

$$x(-A\cos x - B\sin x) - 2A\sin x + 2B\cos x - x(A\cos x + B\sin x) = \sin x$$

$$-2Ax\cos x - 2Bx\sin x - 2A\sin x + 2B\cos x = \sin x$$

نقارن

$$-2A = 1 \Rightarrow A = -1/2$$

$$2B = 0 \Rightarrow B = 0$$

$$\therefore y_p(x) = x\left(-\tfrac{1}{2}\cos x + 0\right) = -\tfrac{1}{2}x\cos x$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{x} + c_2 e^{-x} - \tfrac{1}{2}x\cos x$$