Prepared by Dr. Noorsal Ahmed Zeen Al-Abiden

First Lecture:

Numerical analysis: It is one of the branches of mathematics that focuses on solving complex equations, derivatives, or integrals that are difficult to solve using conventional methods. In numerical analysis, we use iterative methods to reach approximate, highly accurate solutions to mathematical problems.

Example: Find the value of x that satisfies the following equations:

$$2x^2 + 6 = 30$$
$$\Rightarrow 2x^2 = 24$$
$$\Rightarrow x^2 = 12$$
$$\Rightarrow x = 3.4641$$

It is clear that in the above example, the real solution to the equation can be reached using traditional solution methods known in mathematics, so we do not need numerical methods to find the solution.

While in the example below, it is not possible to find the value of x using traditional solution methods due to the presence of the factorial function in the equation:

$$0.4 = \frac{(1-2x)!}{x}$$

As another example, the differentiation of the following function with respect to α cannot be found using the traditional differentiation methods that we studied in calculus books.

$$f(x)=\frac{1}{\beta^{\alpha}(\alpha-1)!}x^{\alpha-1}e^{\frac{x}{\beta}} \quad x,\alpha,\beta>0$$

Before going into the details of the most famous numerical methods used to solve complex mathematical problems, we will study of the errors that occur during the iterative calculation process, as identifying the error in the iterative process is very important for determining the accuracy of the numerical solution.

Sources of errors:

Since numerical solutions are approximate solutions, then there must be a small amount of difference between the Exact solution and the approximate solution. This amount is called error. There are different sources of error in mathematics. We will focus on only two types in this study, which are the errors that have an effective effect. To solve problems using numerical methods.

1- Round- Off Error:-
   This error occurs due to the rotation of numbers during the iterative solution, as decimal places are usually rotated to facilitate the display and reading of numbers. Rounding the decimal places, although small, leads to the accumulation of error after a number of iterations, so the final approximate solution is slightly different from the real solution.

There are some rules that should be followed when rounding decimal numbers, which are as follows:

1-If a number greater than 5 comes after the last decimal place that we want to keep, then we must rotate the number to the top, in other words, delete the extra decimal places and then add the number 1 to the last decimal place that we kept. Example: 9.139. It is rotated two decimal places to the top, 9.14 because The number 3 was followed by 9 (greater than 5). In our example, we have added a number of 0.001 to the original number for the purpose of rounding it.

2- If a number less than 5 comes after the last decimal place that we want to keep, then the number must be rounded to the lowest, in other words, deleting only the extra decimal places without adding anything to the last decimal place that we kept: for example, the number 9.133 is rounded two decimal places to the lowest, 9.13. Because the number 3 was followed by the number 3 (smaller than the number 5), in this example we are subtracting a number of 0.003 from the original number for the purpose of rotation.

3- If a number equal to 5 comes after the last decimal place and we want to keep it, then we must follow the **even number rule**, which has three cases.

• If the number 5 comes after a non-zero number, whether directly or separated by zeros, then the extra decimal places must be deleted and then the number 1 is added to the last decimal place that we kept. Example: The number 8.6251 rotates two decimal places to the top 8.63 because the number 3 came after it. Non-zero, and here the added number is 0.0049. Note that the number 8.62501 also rotates to two decimal places to the top 8.63, meaning that the non-zero number does not have to be directly after the number 5, so the added number in this case is equal to 0.00499.

• If the number 5 is the last non-zero number, and the number is in the last decimal place that we want to keep even, then the number must be rounded to the lowest. Example: The number 8.625 is rounded two decimal places to the lowest 8.62, because the even number 2 came after it only 5, and thus We have subtracted the number 0.005 from the original number for the purpose of completing the rotation.

• If the number 5 is the last non-zero number, and the number is in the last decimal place that we want to keep and increase, then the number must be rotated to the top. Example: The number 8.615 is rotated two decimal places to the lowest 8.62, because the number 1 is odd and only 5 comes after it. Thus, we have collected the number 0.005 from the original number for the purpose of completing the rotation.

2- Truncation error

This error occurs when we truncate a number because it is not possible to write the number completely due to the large number of decimal places or because the interrupted part is not important because it is extremely small.

Example: The Taylor expansion for an exponential function raised to the power of 1 is as follows:

$$e = \sum_{x=0}^{\infty} \frac{1}{x!}$$

$$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots\ldots$$

$$= 2.718282$$

In this case, if the number is cut off at the fourth term, that is, $\frac{1}{3!}$ up to a limit, the value of the exponential function is approximately equal to 2.666667 (note that the solution is given accurately up to six decimal places). But if we cut off at the fifth, that is, $\frac{1}{4!}$ until then, the value of the exponential function is approximately 2.708333, and so on. In fact, the value of the approximate exponential function known above as 2.718282 results from the first ten terms, that is, neglecting the terms after the term $\frac{1}{9!}$.

Types of errors:

Let x it be the true (exact) solution to a specific mathematical problem (for example: $f(x)=0$ the root of the equation, where, by the root of the equation, we mean the value x that makes the function $f(x)$ equal to zero), and let x it be the numerical (approximate) solution to the same problem. Then :

1- **Absolute error**: It is the absolute difference between the real value and the approximate value of the solution $\Delta x = \left| x - x^{*} \right|$, in other words a number: Usually the numerical solution is acceptable if it $\Delta x \le \varepsilon$ is achieved, as ε (read epsilon) is a very small number that is chosen to determine the acceptability and accuracy of the numerical solution (example: $\varepsilon = 0.0001$) .

2-**The relative error**: It is the ratio between the absolute error and the real error, that is $R_{\varepsilon} = \left| \frac{x - x^{*}}{x} \right|$, usually the numerical solution is acceptable if it is achieved $R_{\varepsilon} \le \delta$, as δ (read delta) is a very small number that is chosen to determine the accuracy of the numerical solution. Example: $\delta = 0.0001$

From the above, δ, ε it is clear that both represent the upper limit of the error. It is permissible to occur.

In general, it is clear that in complex mathematical problems we do not know the true solution (x), so it is not possible to find the relative and absolute error using the formulas described above unless we know the true solution. However, we obtain an approximate solution after each iteration using numerical methods. Thus, we can employ the (iterative) numerical solution to find errors.

Let x_n^* it represent the (approximate) numerical solution upon iteration n, since $n = 1, 2, 3 \ldots$. let us assume that the approximate solution will lead to (approaching) the real solution if the number of iterations is large enough. In other words:

$$\lim_{n \to \infty} x_n^* = x$$

Therefore, we can accept the numerical solution if it is achieved $\left| x_n^* - x_{n-1}^* \right| \le \varepsilon$, and if it is achieved, $\left| \frac{x_n^* - x_{n-1}^*}{x_n^*} \right| \le \delta$ since it is the approximate solution resulting from iteration n, and it is the approximate solution resulting from iteration n-1.

Note: In the upcoming lectures, we will dispense with writing an asterisk in the approximate solution symbol when using the two expressions $\left| x_n^* - x_{n-1}^* \right| \le \varepsilon$, $\left| \frac{x_n^* - x_{n-1}^*}{x_n^*} \right| \le \delta$, and for purposes related to ease. Expression. That is, the mathematical expression will be as follows:

$$\left| x_n - x_{n-1} \right| \le \varepsilon, \left| \frac{x_n^* - x_{n-1}^*}{x_n^*} \right| \le \delta.$$

example of absolute and relative errors: Let it be:

$$y = f(x) = x^3 - 6x^2 + 3x - 1.49$$

When , $x = 4.71$

Find the value of the function at, according to the following:

1- Find the real solution up to six decimal places and cut off the rest of the places.

2- The approximate solution when cutting the number is to keep only two decimal places.

3- The approximate solution When rounding the number and keeping only two decimal places..

4- Calculate the absolute and relative error in the second and third requirements compared to the real solution from the first requirement

	x^3	x^2	$6x^2$	$3x$	y
1	104.487111	22.1841	133.1046	14.13	-15.977489
2	104.48	22.18	133.08	14.13	-15.96
3	104.49	22.18	133.08	14.13	-15.95

Absolute and relative error of (2)

$$\Delta y = \left| y - y^* \right|$$

$$= \left| -15.977489 - (-15.96) \right| = 0.017489$$

$$R_y = \left| \frac{y - y^*}{y} \right|$$

$$= \left| \frac{0.017489}{-15.977489} \right| = 0.001094$$

Absolute and relative error of (3)

$$\Delta y = \left| y - y^* \right|$$

$$= \left| -15.977489 - (-15.95) \right| = 0.027489$$

$$R_y = \left| \frac{y - y^*}{y} \right|$$

$$= \left| \frac{0.027489}{-15.977489} \right| = 0.00172$$