

1- Graphs and Graph Models

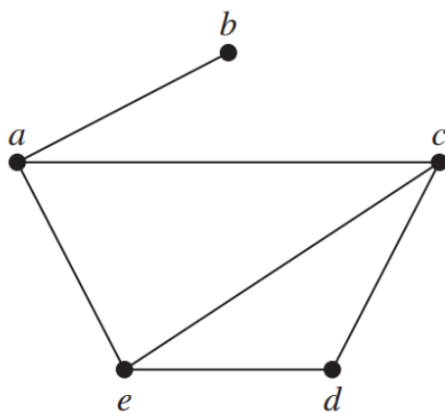
There are many useful ways to represent graphs. In working with a graph, it is helpful to be able to choose its most appropriate representation. In this section, we will show how to represent graphs in several different ways.

1. Representing graph using **adjacency list**.
2. Representing graph using **adjacency matrix**.
3. Representing graph using **incidence matrix**.

1.1 Adjacency Lists

One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

Example 1. Use adjacency lists to describe the following simple graph.

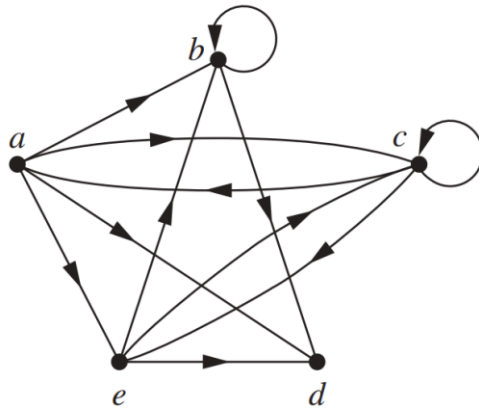


A simple graph.

Solution.

An Adjacency List for a Simple Graph.	
Vertex	Adjacent Vertices
<i>a</i>	<i>b, c, e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a, d, e</i>
<i>d</i>	<i>c, e</i>
<i>e</i>	<i>a, c, d</i>

Example 2. Use adjacency lists to describe the following simple directed graph.



A directed graph.

Solution.

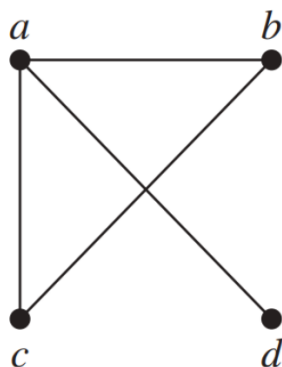
An Adjacency List for a Directed Graph.	
Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

1.2 Adjacency Matrices

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as v_1, v_2, \dots, v_n . The adjacency matrix A (or A_G) of G , with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j) th entry when v_i and v_j are adjacent, and 0 as its (i, j) th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3. Use an adjacency matrix to represent the following graph.

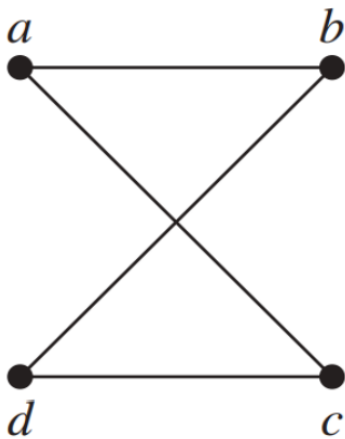


Solution.

$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Note that an adjacency matrix of a graph is based on the **ordering** chosen for the vertices.
Hence, there may be as many as $n!$ different adjacency matrices for a graph with n vertices,
because there are $n!$ different orderings of n vertices.

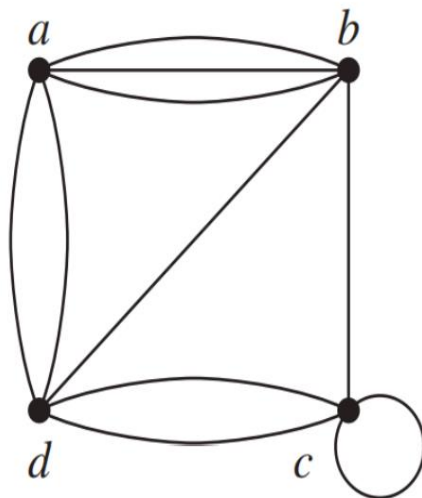
Example 4. Use an adjacency matrix to represent the following graph.



Solution.

$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Example 5. Use an adjacency matrix to represent the following pseudograph.



Solution.

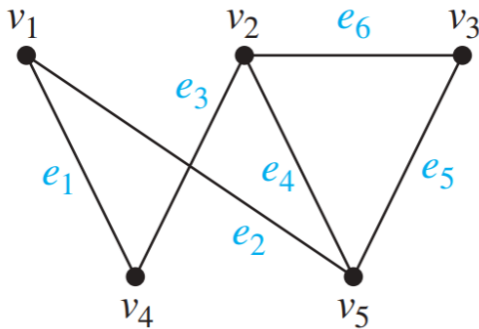
$$\begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$

1.3 Incidence Matrices

Let $G = (V, E)$ be an undirected graph. Suppose that v_1, v_2, \dots, v_n are the vertices and e_1, e_2, \dots, e_m are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

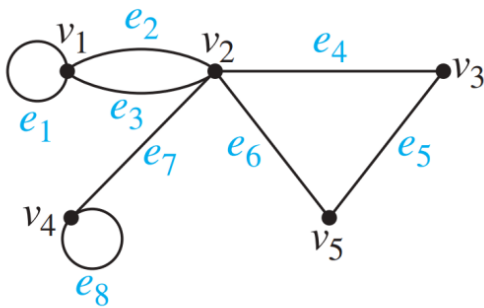
Example 6. Represent the following graph with an incidence matrix.



Solution.

$$\begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Example 7. Represent the pseudograph with an incidence matrix.



Solution.

$$\begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$