Lecture (2): Review in Matrices

Matrix Definition: A set of values or numbers arranged in columns (m) and rows (n). The matrix is symbolized by a capital letter underlined. The matrix contains numbers called elements. Each element is symbolized by the same symbol as the matrix, but it is small letter, i.e. a lowercase letter. Each element is followed by a subscript, the first position of which refers to the row and the second position refers to the column. The values of the matrix, i.e. the elements, are enclosed between parentheses, as follows:

$$\underline{X}_{n*m} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & & & x_{11} \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix}$$

$$i = 1,2, \dots, n$$
$$j = 1,2, \dots, m$$

An example of this is the following matrix:

$$\underline{A}_{2*3} = \begin{bmatrix} 2 & 4 & 3 \\ 5 & 2 & 4 \end{bmatrix} \Longrightarrow \begin{matrix} a_{11} = 2 & a_{12} = 4 & a_{13} = 3 \\ a_{21} = 5 & a_{22} = 2 & a_{23} = 4 \end{matrix}$$

That is, the degree of this matrix is (2*3), meaning that its elements are arranged in two rows and three columns.

Some Types of Matrices

1. **Square Matrix**

   It is the matrix in which the number of rows is equal to the number of columns n=m, and what is called the diagonal.

2. **Diagonal Matrix**

   It is the matrix in which the elements of the diagonal are at least one that is not equal to zero, and the non-diagonal elements are equal to zero. It is always a square matrix, i.e. :

$$a_{ij} = 0 \quad i \neq j$$
$$a_{ij} \neq 0 \quad i = j \qquad \text{One at least}$$

An example of this type of Matrix:

$$\underline{X}_{3*3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here the diagonal is represented by the elements 2, 0, 1, and the remaining zeros represent the non-diagonal elements.

1. **Null Matrix or zero matrix:** All its elements are equal to zero and it is not necessary for it to be square.
2. **Identity Matrix:** It is a diagonal matrix whose elements are equal and equal to one, and it is symbolized by $\underline{\mathrm{I}}_{\mathrm{n}}$ where $\underline{\mathrm{I}}_3$ is an identity matrix with three diagonal elements equal to one.

$$\underline{\mathrm{I}}_{3*3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. **Scalar Matrix:** It is a diagonal matrix in which

$$a_{ij} = K \quad K \neq 0 \; or \; 1$$
$$a_{ij} = 0 \quad i \neq j$$

As an example of this matrix

$$\underline{\mathrm{D}}_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

4. **Symmetric Matrix:** It is a square matrix in which the row elements are equal to the column elements, i.e. $a_{ij} = a_{ji}$ As an example,

$$\begin{bmatrix} 3 & 2 & 5 \\ 2 & 9 & 0 \\ 5 & 0 & 6 \end{bmatrix}$$

5. **Upper Triangle matrix:** It is a square matrix in which the elements of the upper triangle (above the diagonal) are numbers and the elements of the lower third (below the diagonal) are zeros, i.e.

$$a_{ij} \neq 0 \quad j > i$$
$$a_{ij} = 0 \quad j < i$$

An example of this matrix is

$$\underline{A}_{3*3} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Upper triangle

Lower Triangle

6. **Lower Triangle Matrix:** It is a square matrix in which the elements of the lower triangle are numbers and the elements of the upper third are zeros, i.e.

$$a_{ij} \neq 0 \quad j < i \text{ at least one}$$
$$a_{ij} = 0 \quad j > i$$

As an example of this type of Matrices

$$\underline{A} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

7. **Transpose:** This is done by making each row become a column and each column become a row, i.e.: $a_{ij} \rightarrow a_{ji}$, Also, the rank of the matrix changes from n*m to m*n , An example of this is

$$\underline{A}_{2*3} = \begin{bmatrix} 2 & 1 & 0 \\ 5 & 7 & 9 \end{bmatrix} \rightarrow \underline{A}'_{3*2} = \begin{bmatrix} 2 & 5 \\ 1 & 7 \\ 0 & 9 \end{bmatrix}$$

Some properties of Transpose

- $\underline{A}_{n*m} \rightarrow \underline{A}'_{m*n}$
- $(\underline{A}\,\underline{B})' = \underline{B}' * \underline{A}'$
- $(\underline{A}')' = A$
- If $\underline{A}' = \underline{A} \implies \underline{A}$ is symmetric matrix
- $(\underline{A} + \underline{B}) = \underline{B}' + \underline{A}'$
- $\underline{A}'\,\underline{B} = \underline{B}' * \underline{A}$

Vectors

Vector Definition: It is a matrix that contains one row and a number of columns or one column and a number of rows. In this case, the matrix is called a vector, and the vector is symbolized by a lowercase letter underlined.

$$x_{n*1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x'_{1*n} = [x_1 \quad x_2 \quad \cdots \quad x_n]$$

In general, the vector is written in vertical form except in the case of Transpose, in which case it becomes horizontal.

Operations on Matrices

1. **Matrix Addition:** Adding two matrices is adding the corresponding elements of the two matrices, and the rank must be equal. Among the properties of adding matrices are:

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$
$$(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C})$$
$$K(\underline{A} + \underline{B}) = K\underline{A} + K\underline{B}$$

2. **Matrix Multiplication:** Two matrices can be multiplied if the number of columns of the first is equal to the number of rows of the second:

$$A_{\mathrm{n*p}} . B_{\mathrm{p*m}} = C_{\mathrm{n*m}}$$

The element C_{ij} is the sum of the products of the elements of row i of matrix $\underline{A}$ by the corresponding elements in column j of matrix $\underline{B}$.

Matrix Multiplication Properties:

- If it is possible to multiply $\underline{A} * \underline{B}$, it is not necessary that it is possible to multiply $\underline{B} * \underline{A}$.
- It is not necessary that $\underline{A} * \underline{B} = \underline{B} * \underline{A}$.

Vector Multiplication

a) **Transpose pre-multiplication of two vectors:** In this method, one of the vectors must be horizontal and the other vertical, i.e.

$$\underline{\acute{x}}_{1*n}\ \underline{y}_{n*1} = [x_1 \quad x_2 \quad \cdots \quad x_n] . \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^{\mathrm{n}} x_iy_i$$

The above result is called Sum of Cross product or Uncorrected sum of cross product. It can also be found that:

$$\underline{\acute{x}}\ \underline{x} = [x_1 \quad x_2 \quad \cdots \quad x_n] . \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2 = \sum_{i=1}^{\mathrm{n}} x_i^2 \quad \text{uncorrected}$$

sum of squares

b) **Vector Transpost Multiplication:** In this method, it is not necessary for the two vectors to be of the same dimension, but what is important is the inner product.

$$\underline{a}_{n*1} \cdot \underline{\acute{b}}_{1*m} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} * \begin{bmatrix} b_1 & b_2 & \dots & b_m \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_m \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_m \\ \vdots & \vdots & & \vdots \\ a_n b_1 & a_n b_2 & \cdots & a_n b_m \end{bmatrix}_{n*m}$$

We notice that vector $\underline{\acute{b}}$ appeared in a row. The first row is multiplied by $a_1\ x$, the second row is multiplied by $a_2\ x$, and so on to a_n.

Multiply matrix by vector and vector by matrix:

a) Vector Pre-Multiplication:

Here the vector must be taken to be transposed one so that the multiplication operation is possible, as follows:

$$\underline{\acute{a}}_{1*n} \cdot \underline{X}_{n*m} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} . \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}$$

$$= [a_1\ x_{11} + a_2\ x_{21} + \dots + a_n\ x_{n1} \quad a_1\ x_{12} + a_2\ x_{22} + \dots + a_n\ x_{n2} \ \dots\ a_1\ x_{1m} + \dots + a_n\ x_{nm}] = \left[\sum_{i=1}^{n} a_i\ x_{i1} \sum_{i=1}^{n} a_i\ x_{i2} \dots \sum_{i=1}^{n} a_i\ x_{im} \right]_{i*m}$$

b) Post Multiplication:

$$X_{n*m}\ a_{m*1} = \begin{bmatrix} x_{11} a_1 & x_{12} a_2 & \cdots & x_{1m} a_m \\ x_{21} a_1 & x_{22} a_2 & \cdots & x_{2m} m \\ \vdots & \vdots & & \vdots \\ x_{n1} a_1 & x_{n2} a_2 & \dots & x_{nm} a_m \end{bmatrix}_{n*1} = \begin{bmatrix} \sum_{j=1}^{m} a_j\ x_{1j} \\ \sum_{j=1}^{m} a_j\ x_{2j} \\ \vdots \\ \sum_{j=1}^{m} a_j\ x_{nj} \end{bmatrix}$$

Diagonal Matrix Multiplication: This type produces two types of multiplications.

A. **Pre Multiplication:**

$$\underline{D} = \begin{bmatrix} d_{11} & \cdots & \cdots & 0 \\ \vdots & d_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix} \Longrightarrow \begin{bmatrix} x_{11}d_{11} & x_{12}d_{11} & \dots & x_{1m}d_{11} \\ x_{21}d_{22} & x_{22}d_{22} & \cdots & x_{2m}d_{22} \\ \vdots & \vdots & & \vdots \\ x_{n1}d_{nn} & x_{n2}d_{nn} & \dots & x_{nm}d_{nn} \end{bmatrix}_{n*m}$$

B. **Post Multiplication**

$$\underline{X}_{n*m}\,\underline{D}_{m*m} = \begin{bmatrix} x_{11}d_{11} & x_{12}d_{22} & \dots & x_{1m}d_{mm} \\ x_{21}d_{11} & x_{22}d_{22} & \cdots & x_{2m}d_{mm} \\ \vdots & \vdots & & \vdots \\ x_{n1}d_{11} & x_{n2}d_{22} & \dots & x_{nm}d_{mm} \end{bmatrix}_{n*m}$$

We notice in this type of multiplication that each number is multiplied by the corresponding element of $\underline{D}$.

Determinants

The determinant of any matrix is unique and can only be found for a square matrix. If we have a non-irregular matrix (its determinant ≠0), then this determinant is unique and then there is an inverse of the matrix.

Finding Determinant

Determinant of (1*1) Matrix $\underline{A} = [3] \Longrightarrow |\underline{A}| = 3$

Determinant of (2*2) Matrix $\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Longrightarrow |\underline{A}| = a_{11}a_{22} - a_{12}a_{21}$

To find the determinant of a (3*3) matrix, we use the Sarus Diagram. The matrix must be Full Rank, i.e. of the highest rank, i.e. square. This method is used as follows:

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Longrightarrow \underline{A} = \begin{bmatrix} a_{11} & \overset{+}{a_{12}} & \overset{+}{a_{13}} & \overset{+\;-}{a_{11}} & \overset{-}{a_{12}} \\ a_{21} & a_{22} & a_{13} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

$$|\underline{A}| = (a_{11}\,a_{22}a_{33} + a_{12}\,a_{23}a_{31} + a_{13}\,a_{21}a_{32}) \\ - (a_{13}\,a_{22}a_{31} + a_{11}\,a_{23}a_{32} + a_{12}\,a_{21}a_{33})$$

For example, the following matrix:

$$\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \Longrightarrow \underline{A} = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & 1 & 2 \end{bmatrix}$$

$$\therefore |\underline{A}| = [(1*1*2) + (2*3*3) + (3*2*1)] - [(3*1*3) + (1*3*1) + (2*2*2) = 6]$$

Finding Determinants by Finding the Expansion with Minors

For each element of the matrix there is what is called a minor and is symbolized by |M_ij| which is the determinant of the partial matrix remaining from deleting row i and column j in which that element is located

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow |M_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$|M_{33}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Cofactor:

It is the Minor multiplied by a positive or negative sign, depending on the location of the element, as follows:

$$\propto_{ij} = (-1)^{\mathrm{i+j}} * |M_{ij}|$$
$$\propto_{21} = (-1)^{2+1} * |M_{21}| = -|M_{21}|$$
$$\propto_{33} = (-1)^{3+3} * |M_{33}| = |M_{33}|$$

Element factor

Each expanded element is denoted by $\left(EF_{(ij)}\right)$ which is the coefficient $\propto_{ij}$ multiplied by the value of that element:-

$$\left(EF_{(ij)}\right) = a_{ij} \propto_{ij} = \propto_{ij} * (-1)^{i+j} * |M_{ij}|$$

The determinant of any matrix is the sum of the elements of any column or any row of the matrix. It is preferable to choose the row or column that contains zeros to shorten the arithmetic operations because $\left(EF_{(0)} = 0\right)$

Example:

Find the determinant of the following matrix using the method of expansion by minima.

$$\underline{A} = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 6 & 0 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow |\underline{A}| = EF(21) + EF(22) + EF(23) = 5 + 42 + 0 = 47$$

$$|M_{21}| = \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix} = 2 + 3 = 5 \rightarrow \propto_{21} = -1^{2+1}\, |\mathrm{M}_{21}| = -5$$

$$|M_{22}| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = (2 * 2) - (-3 * 1) = 7 \rightarrow \propto_{22} = (-1)^{2+2}\, |\mathrm{M}_{22}| = 7$$

$$\left.\begin{matrix} \Longrightarrow EF(21) = (-1)(-5) = 5 \\ Ef(22) = (6)(7) = 42 \end{matrix}\right\} \Longrightarrow |A| = 42 + 5 = 47$$

Determinants of a matrix of degree (4*4) or more

Here, the same method is followed for the minors in a (3*3) matrix, relying on the row and column that contain values equal to zero.

Example: Find the value of the determinant of the following matrix:

$$\underline{A} = \begin{bmatrix} 2 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{bmatrix}$$

Solution:

We take the third row to find the determinant because it contains two zeros, as follows:

$$|\underline{A}| = 3 * (-1)^{3+1} \begin{bmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{bmatrix} + (-3)(-1)^{3+4} \begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -3 & 4 & 2 & -3 \\ 2 & 1 & 3 & 2 & 1 \\ 0 & -2 & 3 & 0 & -2 \end{vmatrix} = (6+0-16)-(0-12-18) = 20$$

$$=$$

$$\begin{vmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 2 & 1 & -4 & 2 \\ 2 & 0 & -2 & 2 & 0 \end{vmatrix} = (-4+4+0) — 12+0+16$$

$$12-16=-4 \; |\underline{A}| = (3*1*20)+(-3*-1-4) = 60-12 = 48$$

Inverse of Matrix

For some square matrices there is what is called the inverse of the matrix, and the square matrix that has an inverse is called a *non-singular* matrix and must be square, and also when we say that the matrix has an inverse, this means that it has an determinant, and the inverse is unique and the inverse of $\underline{A}$ is symbolized by $\underline{A}^{-1}$

Features of the inverse

1. $\underline{A}\,\underline{A}^{-1} = \underline{A}^{-1}\underline{A}$
2. $\left(\underline{A}^{-1}\right)^{-1} = \underline{A}$
3. $\left(\underline{\acute{A}}\right)^{-1} = (\underline{A}^{\acute{-1}})$
4. $(\underline{A}\,\underline{B})^{-1} = \underline{B}^{-1}\,\underline{A}^{-1}$
5. $|A^{-1}| == \frac{1}{|A|}$
6. If $\underline{\acute{A}} = \underline{A}^{-1} \Longrightarrow \underline{\acute{A}}\,.\,\underline{A}^{-1} = I \Longrightarrow$ $A \text{ is or thogonal matrix}$
7. let $\underline{D} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & & 0 \\ 0 & \frac{1}{d_{22}} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{d_{mm}} \end{bmatrix} \Longrightarrow \underline{D}^{-1} = \begin{bmatrix} d_{11} & 0 & & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & & & d_{mm} \end{bmatrix}$

That is, if we have a diagonal matrix, the inverse of this matrix $\underline{D}^{-1}$ is the reciprocal of the elements of the diagonal of this matrix.

Inverse from the Adjoint

If the reduced element $|M_{ij}|$ is the determinant of the matrix remaining after deleting row *i* and column *j* and

$\propto_{ij} = (-1)^{i+j} * |M_{ij}|$ is the reduced element multiplied by a positive or negative sign depending on the location of that element.

Adjoint

It is about converting all its elements into a cofactor and then taking the transpose for that matrix, i.e.

$$\propto_{ij} \longrightarrow \propto_{ji}$$

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow Adj\ \underline{A} = \begin{bmatrix} \propto_{11} & \propto_{12} & \propto_{13} \\ \propto_{21} & \propto_{22} & \propto_{13} \\ \propto_{31} & \propto_{32} & \propto_{33} \end{bmatrix} \Longrightarrow \underline{A}^{-1} = \frac{1}{|\underline{A}|}(adj\ \underline{A})$$

Example: Find the inverse of the following matrix using the *adj*, where

$$\underline{A} = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

Solution:

$$|\underline{A}| = -94$$

$$\propto_{11} = (-1)^{1+1} . \begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18 \qquad \propto_{23} = (-1)^{2+3} . \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2$$

$$\propto_{12} = (-1)^{1+2} . \begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17 \qquad \propto_{31} = (-1)^{3+1} . \begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10$$

$$\propto_{13} = (-1)^{1+3} . \begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6 \qquad \propto_{32} = (-1)^{3+2} . \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1$$

$$\propto_{21} = (-1)^{2+1} . \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6 \qquad \propto_{32} = (-1)^{3+3} . \begin{vmatrix} 7 & -2 \\ 5 & 6 \end{vmatrix} = 28$$

$$\propto_{22} = (-1)^{2+2} . \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10$$

$$\therefore Adj\ \underline{A} = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} \Longrightarrow \underline{A}^{-1} = \frac{-1}{94}\ adj\ \underline{A}$$

$$= \begin{bmatrix} 0.19 & 0.064 & 0.11 \\ -0.181 & 0.11 & 0.01 \\ 0.06 & 0.02 & 0.3 \end{bmatrix}$$

Finding the Inverse Using Doolittle Method

This method is used when we have a symmetric matrix, and is widely used in multiple regression analysis.

Example: *by Doolittle Find the inverse of* $X = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 10 & 16 \\ 6 & 16 & 30 \end{bmatrix}$

L.N	Original Matrix	I	Sum
1	2 4 6	1 0 0	13
2	10 16	0 1 0	31
3	30	0 0 1	53
4	2 4 6	1 0 0	13
5	1 2 3	0.5 0 0	6.5
6	2 4	−2 1 0	5
7	1 2	−1 0.5 0	2.5
8	4	1 −2 1	4
9	1	0.25 −0.5 0.25	1

The last row represents the last column in $\underline{X}^{-1}$.

$$C_{11} = (1)(0.5) + (-2)(-1) + (1)(0.25) = 2.75$$
$$C_{12} = (1)(0) + (-2)(0.5) + (1)(-0.5) = -1.5$$
$$C_{13} = (1)(0) + (-2)(0) + (1)(0.25) = 0.25$$
$$C_{22} = (0)(0) + (1)(0.5) + (-2)(-0.5) = 1.5$$
$$C_{23} = (0)(0) + (1)(0) + (-2)(0.25) = -0.5$$
$$C_{33} = (0)(0) + (0)(0) + (1)(0.25) = 0.25$$

The inverse is symmetric because the determinant is symmetric.

$$\therefore \underline{X}^{-1} = \begin{bmatrix} 2.75 & -1.5 & 0.25 \\ & 1.5 & -0.5 \\ & & 0.25 \end{bmatrix}$$

Example: by Doolittle Find the inverse of A

$$\underline{A} = \begin{bmatrix} 2 & 8 & -4 & 0 \\ 8 & 26 & 4 & 1 \\ -4 & 4 & 12 & 0 \\ 0 & 1 & 0 & 10 \end{bmatrix}$$

L. N	Original Matrix	I	Sum
1	2 8 −4 0	1 0 0 0	7
2	8 26 4 1	0 1 0 0	40
3	−4 4 12 0	0 0 1 0	13
4	0 1 0 10	0 0 0 1	12
5	2 8 −4 0	1 0 0 0	7
6	8 26 4 1	0.5 0 0 0	3.5
7	0 −6 20 1	−4 1 0 0	12 A_1
8	0 1 −3.333 −0.166	0.6667 −0.1667 0 0	−2
9	0 0 70.667 3.334	−11.334 3.334 1 0	67 B_1
10	0 0 1 0.0473	−1.1604 0.0472 0.0142 0	0.948
11	0 0 0 10.0094	−0.132 −0.0094 −0.0673 1	10.8396 C_1
12	0 0 0 1	−0.0132 −0.009 0.0047 0.09	1.0082
		العمود الاخير في $\underline{A}^{-1}$	

$A_{11} = 26 - (8 * 4) = -6$

$A_{12} = 4 - (8 * -2) = 20$

$A_{13} = 1 - (8 * 0) = 1$

$A_{14} = 0 - (8 * 0.5) = -4$

$A_{15} = 1 - (8 * 0) = 1$

$A_{16} = 0 - (8 * 0) = 0$

$A_{17} = 0 - (8 * 0) = 0$

$A_{18} = 40 - (8 * 3.5) = 12$

$B_{11} = 12 — 4 * -2 + (20 * -3.333) = 0.667$

$B_{12} = 0 - (- - 4 * 0 + 20 * -0.1667) = 3.334$

$B_{13} = 0 — 4 * 0.5 + 20 * 0.6667 = -11.334$

$B_{14} = 0 — 4 * 0 + 20 * -0.1667 = 3.334$

$B_{15} = 1 — 4 * 0 + (20 * 0) = 1$

$B_{16} = 0 — 4 * 0 + (20 * 0) = 0$

$B_{17} = 13 — 4 * 3.5 + 20 * -2 = 67$

$C_{11} = 10 - (0 * 0 + 1 * -0.1667 + 3.334 * 0.00472) = 10.0094$

$C_{12} = 0 - (0 * 0.5 + 1 * -0.1667 + 3.334 * -0.1604) = -0.132$

$C_{13} = 0 - (0 * 0 + 1 * -0.1667 + 3.334 * 0.0472) = -0.0094$

$C_{14} = 0 - (0 * 0 + 1 * 0 + 3.334 * 0.0142) = -0.0473$

$C_{15} = 1 - (0 * 0 + 1 * 0 + 3.334 * 0) = 1$

$$\therefore \underline{A}^{-1} = \begin{bmatrix} -0.346 & 0.1329 & -0.1602 & -0.0132 \\ & -0.0002 & 0.0473 & -0.0009 \\ & & 0.0144 & -0.0047 \\ & & & 0.0009 \end{bmatrix}$$

$a_{11} = (1 * 0.5 \pm 4 * 0.667 \pm 11.334 * -0.1204 \pm 0.132 * -0.0132) = -0.346$

$a_{12} = 1 * 0 \pm 4 * 0.16670 - 11.334 * 0.047 \pm 0.132 * -0.0009 = 0.1329$

$a_{13} = 1 * 0 \pm 4 * 0 \pm 11.334 * 0.0142 \pm 0.132 * -0.00047 = -0.1602$

$a_{21} = 0 * 0 + 1 * -0.1667 + 3.334 * 0.0472 \pm 0.094 * -0.009 = -0.0092$

$a_{23} = 0 * 0 + 1 * 0 + 3.334 * 0.0142 \pm 0.0094 * -0.0047 = 0.0473$

$a_{33} = 0 * 0 + 0 * 0 + 1 * 0.0142 \pm 0.0473 * -0.0047 = 0.0144$