In the previous lecture, we previously discussed the definition of absolute error as the absolute difference between the true value x and the approximate value x^*, meaning: In the same way $\Delta x = \left| x - x^* \right|$,

the relative error is $R_\varepsilon = \left| \dfrac{x - x^*}{x} \right|$

Example 1: Let $x = 5.267$ represent the real value and $x^* = 5.57$ represent the approximate value, find the absolute and relative errors.

Example 1: find the absolute and relative errors

Solution: Absolute error:

$$\Delta x = \left| x - x^* \right| = \left| 5.267 - 5.27 \right| = 0.003 = 3 \times 10^{-3}$$

Relative error

$$R_\varepsilon = \left| \frac{x - x^*}{x} \right| = \left| \frac{5.267 - 5.27}{5.267} \right| = \left| \frac{0.003}{5.267} \right| = 0.000569$$

Example: 2 (about rounding errors): Round the number $x = 9.3565$ to three decimal places, then to two decimal places, then to one decimal place, and determine the upper limit of the absolute error committed in the rotation process for each case.

The solution:

Rotate to three decimal places $x^* = 9.357$, and the upper limit The absolute error committed is $\varepsilon = 5 \times 10^{-4}$, 0.0005

rounded to two decimal places $x^* = 9.36$, and the upper limit of the absolute error committed is $\varepsilon = 4 \times 10^{-3}$, 0.004

rounded to one decimal place $x^* = 9.4$, and the upper limit of the absolute error committed is $\varepsilon = 4 \times 10^{-2}$, 0.04

Trigonometric inequality for absolute value:

The absolute value of the total of the two real numbers (x, y) is less than or equal to the sum of the absolute values of the two numbers, that is, that:

$|x + y| \le |x| + |y|$ .... (1)

The stabilizers are not required.

It can be reached to the fact that the absolute difference between two real numbers is less than or equal to the sum of the absolute values of the two numbers, and that is: that:

$|x - y| \le |x| + |y|$......................... (2)

And for this, put $y = -z$ in the equation 1 above:

$$|x + (-z)| \le |x| + |-z|$$

$$\Rightarrow |x - z| \le |x| + |z|$$

Errors in Arithmetic Operations:

Let's assume that y, x represent the true values, and y^*, x^* represent the approximate values. If we want to perform some algebraic operations (addition, subtraction, multiplication, division) on these approximate numbers, the error in the result of the algebraic operation will be subject to the following rules:

1. **Error in Addition and Subtraction:**

The absolute error in the sum (or difference) of two approximate numbers is less than or equal to the sum of the absolute errors of these two numbers.

Proof for Addition:

Let $u^* = x^* + y^*$ represent the approximate values of the numbers $u = x + y$, respectively. The error can be written as:

$$u - u^* = (x + y) - (x^* + y^*)$$
$$\Rightarrow u - u^* = (x - x^*) + (y - y^*)$$
$$\Rightarrow \left|u - u^*\right| = \left|(x - x^*) + (y - y^*)\right|$$

Applying the triangle inequality for absolute values to the right-hand side, we get:

$$\left|u - u^*\right| \leq \left|(x - x^*) + (y - y^*)\right|$$
$$\Rightarrow \Delta u \leq \Delta x + \Delta y$$

The last inequality means that the absolute error in the result of adding two approximate numbers is less than or equal to the sum of the absolute errors of the two numbers being added.

Proof for Subtraction:

Let $v^* = x^* - y^*$ represent the approximate values of $v = x - y$, respectively. The error can be expressed as:

$$\Delta v = \left| v - v^* \right|$$

$$\Rightarrow v - v^* = (x - y) + (\mathrm{x}^* - y^*)$$

$$\Rightarrow v - v^* = (x - \mathrm{x}^*) - (y - y^*)$$

$$\Rightarrow \left| v - v^* \right| = \left| (x - \mathrm{x}^*) - (y - y^*) \right|$$

By applying the triangle inequality for absolute values to the difference between two real numbers, we get:

$$\Rightarrow \left| v - v^* \right| \leq \left| (x - \mathrm{x}^*) - (y - y^*) \right|$$

$$\Rightarrow \Delta v \leq \left| \Delta x \right| + \left| \Delta y \right|$$

This means that the absolute error in the result of subtracting two approximate numbers is less than or equal to the sum of the absolute errors of the two numbers being subtracted.

Example: Assume that $x^* = 3.221$ and $y^* = 3.222$ are rounded numbers (i.e., they originally had decimal places which were rounded). The task is to find the upper bound of the absolute error for the quantities $u = x + y$ and $v = x - y$.

Solution: Since the numbers mentioned in the question are rounded, there must be an absolute rounding error not exceeding 0.0005, or 5×10^{-4} in other words:

$$\Delta x \leq 5 \times 10^{-4}$$

$$\Delta y \leq 5 \times 10^{-4}$$

Thus:

$$\Delta x + \Delta y \le 5\times10^{-4} + 5\times10^{-4} = 10\times10^{-4} = 0.0010$$

According to the error propagation rule for addition:

$$\Delta u \le \Delta x + \Delta y$$

$$\Delta u \le 0.0010$$

Similarly, it can be concluded that:

$$\Delta v \le \Delta x + \Delta y$$

$$\Delta v \le 0.0010$$

2. **Error in Multiplication:**

Let $s^* = x^* y^*$ represent the approximate values of the numbers $s = x\ y$, Then the absolute and relative errors for the product are:

$$\Delta s \le \left|x^* \Delta y\right| + \left|y^* \Delta x\right|$$

$$R_S \le R_x + R_y$$

Example: Find the absolute error for the product of the approximate numbers $y^* = 0.281,\ x^* = 3.7148$

Solution: Given that the numbers are approximate, the upper bound for the absolute error in the product is:

$$\Delta x \le 5\times10^{-5}$$

$$\Delta y \le 5\times10^{-4}$$

Thus, the absolute error for the product is:

$$\Delta s \leq \left| x^* \Delta y \right| + \left| y^* \Delta x \right|$$

$$\Delta s \leq 3.7148 \times 0.0005 + 0.281 \times 0.00005$$

$$= 0.001857 + 0.000014 = 0.001871$$