factorials

Factorials values are used extensively in number theory and for approximations. Factorials are denoted by the exclamatory mark (!).

What is a Factorial?

The use of ! was started by Christian Kramp in 1808. In mathematics, the factorial of a non-negative integer n, denoted by $n!$ is the product of all positive integers less than or equal to n. Where n can be any natural number. The factorial of n.

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

Thus $\text{n}! = \prod_{i=0}^{n} n \ldots\ldots\ldots\ldots\ldots\ldots\ldots. n \geq 0$

The expression $n!$ is read “n factorial”.

This definition is extended to number 0 by using the convention:

$$0! = 1$$

Two Ways to Expand the Factorial of the Variable n Written as n!

Descending Order

The factorial of a number n or n! can be written as follows in descending order

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

Ascending Order

The factorial of a number n or n! can be written as follows in ascending order

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-3) \times (n-2) \times (n-3)$$

Uses of Factorials

Factorials are used in the following Applications

In probability theory, there are many scenarios in which we must calculate all the possible arrangements of a given set. In probability theory, factorials are extensively used in the evaluation of permutations and combinations.

For example, if we toss a coin 20 times, what would be the size of sample space.

Or if we want to select a team of 10 students from a class of 50 members, how many different teams can we make? Factorial is a mathematical operation that helps us in figuring out such arrangements and hence plays an important role in probability theory.

1. The factorial rule simplifies the calculations of complex expressions involving factorials.
2. We may use factorial in the recursive definition of a number.
3. Probability distributions like binomial distribution include the use of factorial, to find the probability of an event.
4. Factorials values are used extensively in number theory and for approximations.

Solved Examples

1. What is the factorial of 5?

The numbers less than or equal to 5 are 5, 4,3,2,1. Therefore,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

2. Evaluate $\frac{7!}{4!}$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$4! = 4 \times 3 \times 2 \times 1$$

Therefore $\frac{7!}{4!} = \frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

$$= (7 \times 6 \times 5)$$

$$= 210$$

3. Evaluate $\frac{6!}{3\times4!}$

Using the factorial rule, we can write

$$6! = 6 \times 5\, x\, 4\, x\, 3\, x\, 2\, x\, 1$$

However, we have 4! in the denominator, so we apply the factorial rule again on this to get

$6! = 6 \times 5 \times 4!$. Accordingly,

$$\frac{6!}{3 \times 4!} = \frac{6 \times 5 \times 4!}{3 \times 4!} = \frac{6 \times 5}{3} = 10$$

4. Evaluate $\frac{11!}{5! \times 8!}$

$$11! = 11 \times 10 \times 9 \times 8!$$

Therefore,

$$\frac{11!}{5! \times 8!} = \frac{11 \times 10 \times 9 \times 8!}{5! \times 8!} = \frac{11 \times 10 \times 9}{5!} = \frac{990}{120} = 8.25$$

Factorials Properties of Zero

The definition of the factorial states that the value of $\mathbf{0! = 1}$. Factorial is the product of all integers equal to or less in value to the original number. A factorial is the number of combinations possible with numbers less than or equal to that number. Zero has no numbers less than it but is still in and of itself a number.

If we want 1 to follow the factorial rule, then from the formula of the factorials, it is obvious that 1! = 1.

However, by factorial rule,

$$1! = 1 \times 0!$$

$$1 = 1 \times 0!$$

This means that $0! = 1$. Factorial is not defined for negative integers, and hence factorial rules cannot be applied on 0.

Hence, by definition, the factorial of zero is 1.

Factorials Properties

The properties of factorials are as follows:

1- $n! = n \times (n-1)!$

2- $(n-1)! = \frac{n!}{n}$

3- $n! = \prod n = \int_0^1 (-\ln t)^x dt = \int_0^\infty t^x e^{-t} dt, x > -1$ gives the factorial of x for all real positive numbers. This is known as the Bernoulli interpolating function of factorials

4- $\prod(x+1) = (x+1)\prod(x), x \geq 0$ This is known as the Euler's factorial function. It also known as the Pi function

5- $\gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ This is known as the Euler′s gamma function. It is related to the Pi functions

Example: In how many ways can 6 people be arranged in a row?

Solution: $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

Sampling with replacement: - selected subjects are put back into the population before another subject is sampled. Subject can possibly be selected more than once.

Sampling without replacement: - Selected subjects will not be in the "pool" for selection. All selected subjects are unique. This is the default assumption for statistical sampling.

Example: With 6 names in a bag, randomly select a name. How many ways can the 6 names be assigned to 6 job assignments?

Solution: A) Sampling without replacement occurs when an object is not replaced after it has been selected $6.5.4.3.2.1 = 6! = 720$

B) Sampling with replacement occurs when an object is selected and then replaced before the next object has been selected $6.6.6.6.6.6 = 46656$

Permutations

The arrangement of $\boldsymbol{n}$ objects in a specific order using $\boldsymbol{r}$ objects at a time is called a permutation of $\boldsymbol{n}$ objects taking $\boldsymbol{r}$ objects at a time. It is written as $\boldsymbol{nPr}$, and the formula is

$$\boldsymbol{nP_r = p(n,r) = \frac{n!}{(n-r)!}}$$

Where

n = number of objects

r = number of positions

Example:

1) $p(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5 \times 4 \times 3 \times 2 \times 1 = 120$
2) $p(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 6 \times 5 \times 4 \times 3 = 360$

Example: A student has 10 books to arrange on a shelf of a bookcase. In how many ways can the 10 books be arranged?

Solution:

$$p(10,10) = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= 3{,}628{,}800$$

The Special Permutations

Rule The number of possible permutations of m objects among themselves is $m!$.

$$p(m,m) = \frac{m!}{(m-m)!} = m!$$

Combinations

The number of combinations of r objects selected from n objects is denoted by $nCr, \binom{n}{r}, C_r^n$ and is given by the formula

$$C_r^n = \frac{n!}{r!\,(n-r)!}$$

Note: Combinations are used when the order or arrangement is not important, as in the selecting process. Suppose a committee of 5 students is to be selected from 25 students. The five selected students represent a combination, since it does not matter who is selected first, second, etc.

Example: How many combinations of 4 objects are there, taken 2 at a time?

Solution: $n = 4\,, r = 2$

$$C_2^4 = \frac{4!}{2!\,(4-2)!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

Example: A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

Solution: n=8 ,=3

$$C_3^8 = \frac{8!}{3!\,(8-3)!} = \frac{8!}{3!\,5!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{3\times2\times1\times5\times4\times3\times2\times1} = 56$$

Probability and permutations

Probability, as you learned in a previous section, has the formula

$$probability\ (\text{success}) = \frac{\text{number of way to get succes}}{total\ number\ of\ possible\ outecome}$$

Example: The word COUNTED has been spelled using scrabble tiles. Two tiles are randomly chosen one at a time and placed in order in which they were chosen. Determine the probability that the tiles are:

Solution:

a) $P(CO) = \frac{1}{p_2^7} = \frac{1}{42} = 0.0238$

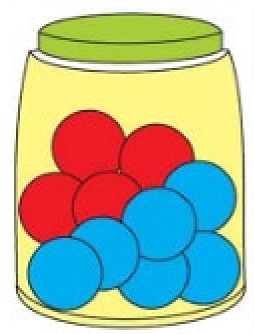
b) $P(both\ vowels) = \frac{p_2^3}{p_2^7} = \frac{6}{42} = 0.143$

Probability and Combinations:

Combinations are used in probability when the order does not matter.

$$\text{probability (success)} = \frac{\text{number of way to get succes}}{total\ number\ of\ possible\ outcomes}$$

Example: Suppose you have ten marbles: four red and six blue. You choose three marbles without looking. What is the probability that all three marbles are red?



Solution: there are C_3^4 ways to choose the red marbles.

There are C_3^{10} total combinations.

$$\text{probability (success)} = \frac{\text{number of way to get succes}}{total\ number\ of\ possible\ outcomes}$$

$$\text{probability (all 3 marble are red)} = \frac{C_3^4}{C_3^{10}} = \frac{4}{120} = \frac{1}{30}$$

Example: Two cards are selected without replacement from a desk 52 playing cards. Determine the probability that both cards are kings using combinations.

Solution: There are C_2^4 ways to drawing 2 kings.

There are C_2^{52} total ways of drawing 2 cards.

$$\text{probability (2 king)} = \frac{C_2^4}{C_2^{52}} = \frac{6}{1326} = 0.0045$$

Example:

a) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if order is not important?

Solution: n=10 ,=4

$$C_4^{10} = \frac{10!}{4!\,(10-4)!} = 210$$

H.W.

a) How many ways are there to choose a committee of 4 persons from a group of 10 persons, if one is to be the chairperson?

b) What is the probability that a particular person is not on the committee?