The Continuous Fourier Transform

Let $f(x)$ be a continuous function of a real variable x. The Fourier transform of $f(x)$, denoted by $\Im\{f(x)\}$, is defined by the equation :

$$\Im\{f(x)\} = \mathrm{F(u)} \underline{\Delta} \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx \qquad \ldots\ldots\ldots.(1)$$

Where $j = \sqrt{-1}$

Given $F(u)$, $f(x)$ can be obtained by using the inverse Fourier transform:

$$\Im^{-1}\{F(u)\} = \mathrm{f(x)} \underline{\Delta} \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du \qquad \ldots\ldots\ldots.(2)$$

Equations (1) and (2) , which are called the Fourier transform pair, can be shown to exist if $f(x)$ is continuous and integrable and $F(u)$ is integrable. These conditions are almost always satisfied in practice.

The Fourier transform of a real function, however is generally complex, that is:

$$F(u) = R(u) + jI(u) \qquad \ldots\ldots\ldots\ldots(3)$$

where *R(u)* and *I(u)* are respectively, the real and imaginary components of *F(u).* It is often convenient to express Eq. (3) in exponential form:

$$F(u) = |F(u)|\ e^{j\phi(u)}$$

where

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

and

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

The magnitude function $|F(u)|$ is called the Fourier spectrum of $f(x)$; and $\phi(u)$ its phase angle. The square of the spectrum:

$$E(u) = |F(u)|^2$$
$$= R^2(u) + I^2(u)$$

is commonly referred to as the energy spectrum of $f(x)$.

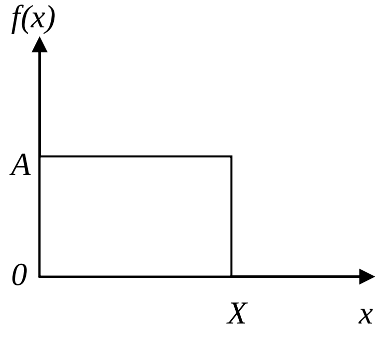
The variable u appearing in the Fourier transform is often called the frequency variable, This name arises from the fact that using Euler's formula the exponential term $\exp[-j2\pi ux]$ may be expressed in the form:

$$\exp[-j2\pi ux] = \cos 2\pi ux - j\sin 2\pi ux$$

Example: consider the simple function shown in the following figure. Its Fourier transform is obtained from Eq(1) as follows:

$$\mathrm{F(u)} = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi\, ux]dx$$

$$= \int_{0}^{X} A\exp[-j2\pi\, ux]dx$$



$$= \frac{-A}{j2\pi u}\left[e^{-j2\pi ux}\right]_0^X$$

$$= \frac{-A}{j2\pi u}\left[e^{-j2\pi uX} - 1\right]$$

$$= \frac{A}{j2\pi u}\left[e^{j\pi uX} - e^{-j\pi uX}\right]e^{-j\pi uX}$$

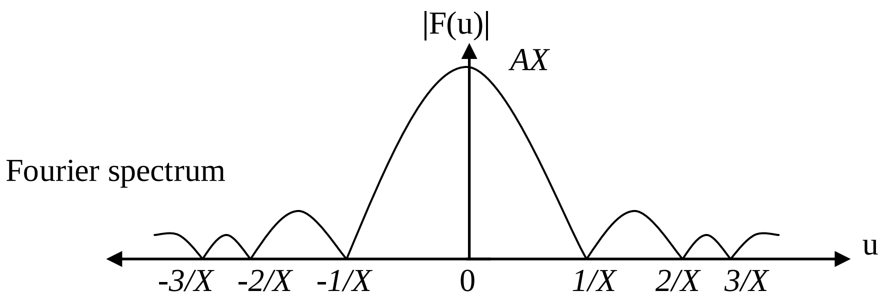
$$= \frac{A}{\pi u}\sin(\pi uX)\, e^{-j\pi uX}$$

which is a complex function . The Fourier spectrum is given by:

$$|\mathrm{F(u)}| = \left|\frac{A}{\pi u}\right| \left|\sin(\pi uX)\right| \left|e^{-j\pi uX}\right|$$

$$= AX\left|\frac{\sin(\pi uX)}{(\pi uX)}\right|$$

A plot of $|F(u)|$ is shown in this fig.



The Fourier transform can be easily extended to a function $f(x,y)$ of two variables. If $f(x,y)$ is continuous and integrable and $F(u,v)$ is integrable, we have that the following Fourier transform pair exists:

$$\Im\{\, f(x,y)\,\} = \mathrm{F(u,v)} \underline{\underline{\Delta}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

and

$$\Im^{-1}\{\, F(u,v)\,\} = f(x,y) \underline{\underline{\Delta}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[\,j2\pi(ux+vy)]dudv$$

Where *u* and *v* are the frequency variables.

As in the one-dimensional case, the Fourier spectrum, phase, and energy spectrum are, respectively, given by the relations:

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

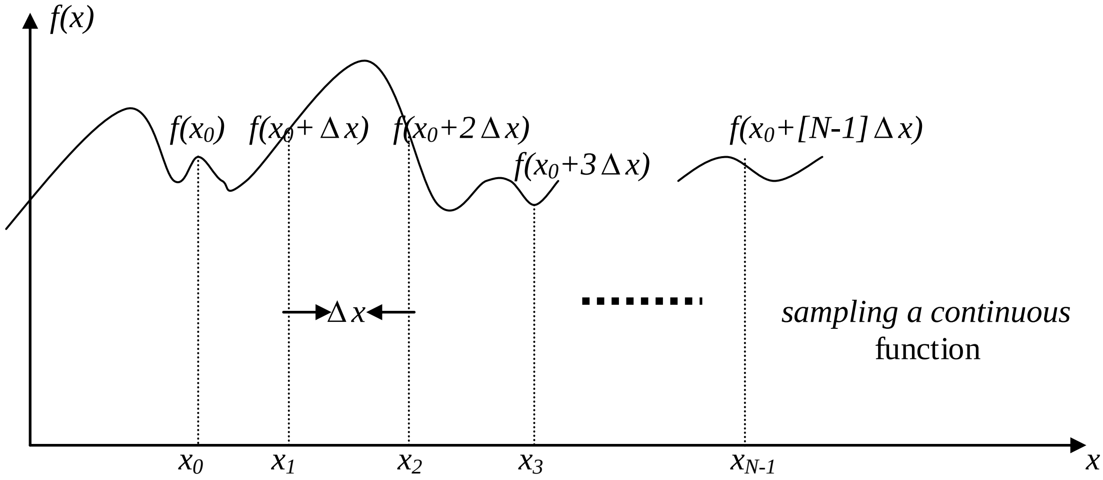
and

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

$$E(u,v) = R^2(u,v) + I^2(u,v)$$

The Discrete Fourier Transform

Suppose that a continuous function $f(x)$ is discretized into a sequence $\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots\ldots\ldots, f(x_0+[N-1]\Delta x)\}$ by taking N samples Δx units a part, as shown in the following figure. It will be convenient in subsequent developments to use x as either a discrete or continuous variable, we may do this by defining: $f(x) = f(x_0 + x\Delta x)$



sampling a continuous function

Where x now assumes the discrete values *0,1,2 , ... , N-1* . In other words the sequence $\{f(0), f(1), f(2), \ldots\ldots\ldots, f(N-1)\}$ will be used to denote any N uniformly spaced samples from a corresponding continuous function. The discrete fourier transform pair that applies to sampled functions is given by:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi\, ux/N] \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

for $u=0,1,2, \ldots, N-1$ and

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi\, ux/N] \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

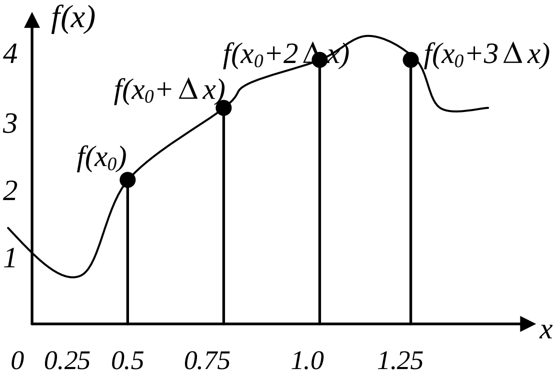
for $x=0,1,2 , \ldots, N-1$

The values $u=0,1,2, \ldots ,N-1$ in the discrete fourier transform given in Eq. (1) correspond to samples of the continuous transform at values $0, \Delta u, 2\Delta u, \ldots, (N-1)\Delta u$

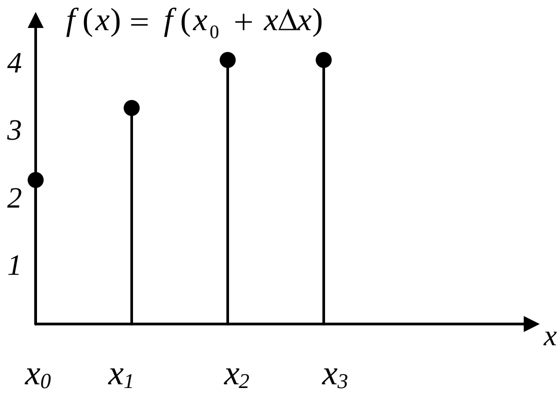
In other words we are letting *F(u)* represent $F(u\Delta u)$. This notation is similar to that used for the discrete *f(x)*, with the exception that the samples of

$F(u)$ start at the origin of the frequency axis. It can be shown that Δu and Δx are related by the expression: $\Delta u = \dfrac{1}{N\Delta x}$

Example :
As an illustration of Eqs (1) and (2) consider the function shown in figure (a). If this function is sampled at the argument values x_0=0.5 , x_1=0.75 , x_2=1.0 , x_3=1.25 and if the argument is redefined as discussed above, we obtain the discrete function shown in fig. (b).



(a) x is continuous



(b) x is discrete

Application of Eq. (1) to the resulting four samples yields the following sequence of steps:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi\, ux/N]$$

$$F(0) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi\, x*0/4]$$

$$\text{F(0)} = \frac{1}{4}\sum_{x=0}^{3} f(x)\, e^{0}$$

$$= \frac{1}{4}\ [\ f(0) + f(1) + f(2) + f(3)\ ]$$

$$= \frac{1}{4}\ [\ 2 + 3 + 4 + 4\ ]$$

$$= 3.25$$

$$F(1) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j2\pi\, x*1/4]$$

$$= \frac{1}{4}[\ 2\ e^{0}\ + 3\ e^{-j\pi/2}\ + 4\ e^{-j\pi}\ + 4\ e^{-j3\pi/2}]$$

$$= \frac{1}{4}[2 + 3\ (\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}) + 4\ (\cos\pi - j\sin\pi) + 4\ (\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2})]$$

$$= \frac{1}{4}\ [2 + 3\ (0 - j) + 4\ (-1 - 0) + 4\ (0 + j)]$$

$$= \frac{1}{4}\ [\ 2 - 3j - 4 + 4j\ ]$$

$$= \frac{1}{4}[\,-2 + j\,]$$

$$F(2) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j4\pi\ x/4]$$

$$= \frac{1}{4}[\ 2\ e^{0}\ + 3\ e^{-j\pi}\ + 4\ e^{-j2\pi}\ + 4\ e^{-j3\pi}\,]$$

$$= -\ \frac{1}{4}[1 + 0\,j\,]$$

and

$$F(3) = \frac{1}{4}\sum_{x=0}^{3} f(x)\ \exp[-j6\pi\ x/4]$$

$$= \frac{1}{4}[\ 2\ e^{0}\ + 3\ e^{-j3\pi/2}\ + 4\ e^{-j3\pi}\ + 4\ e^{-j9\pi/2}\,]$$

$$= -\ \frac{1}{4}[2 +\ j\,]$$

The Fourier spectrum is obtained from the magnitude of each of the transform terms, that is :

$|F(0)| = 3.25$

$|F(1)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4$

$|F(2)| = [(1/4)^2 + (0/4)^2]^{1/2} = 1/4$

$|F(3)| = [(2/4)^2 + (1/4)^2]^{1/2} = \sqrt{5}/4$

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