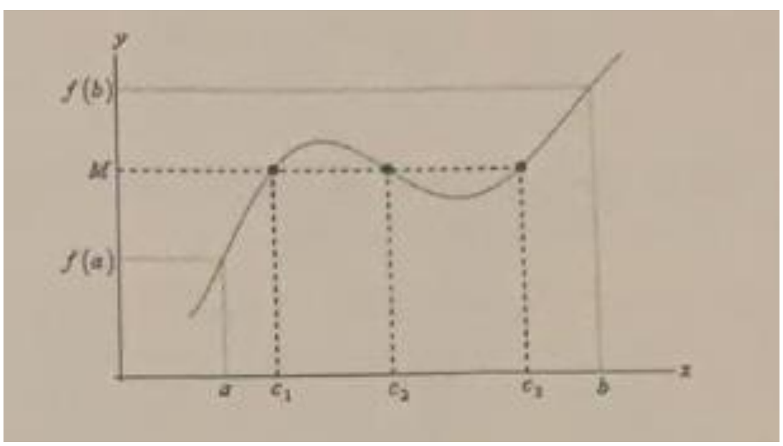
Lecture 4: Methods for Solving Nonlinear Equations

In the previous lecture, we discussed how to estimate the roots of nonlinear equations using the graphical method. However, this method lacks accuracy in pinpointing the exact value of the roots. Therefore, it is necessary to rely on more accurate and objective methods to estimate the roots of nonlinear equations. Before delving into these methods, we will first explore some important mathematical theorems to better understand numerical methods.

Intermediate Value Theorem:

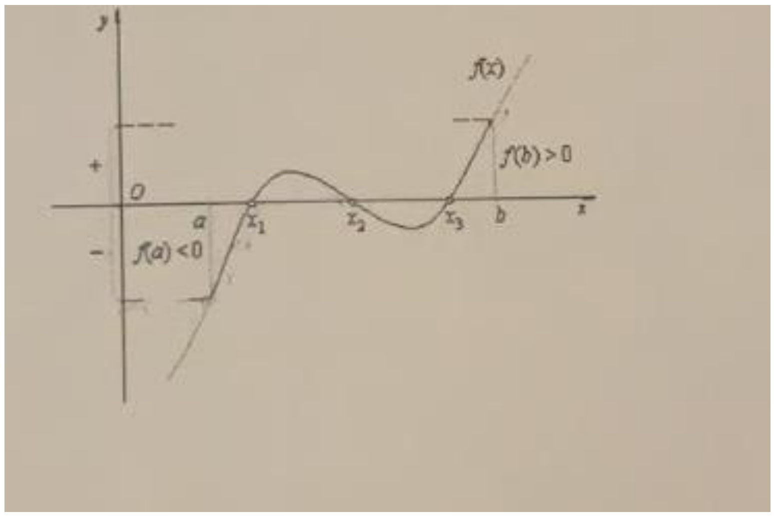
Let $f(x)$ be a continuous function defined on a closed interval $[a,b]$. Let m be any value between $f(\mathrm{a}), f(b)$, meaning $f(\mathrm{a}) < \mathrm{m} < f(b)$. Then, there must be at least one number c such that $a < c < b$ and $f(\mathrm{c}) = \mathrm{m}$.



Bolzano's Theorem:

Let $f(x)$ be a continuous function defined on a closed interval $[a,b]$. If $f(\mathrm{a}), f(b) < 0$, , then there must be at least one number c such that $a < c < b$, and $f(c) = 0$

In other words, if the product of the values of the function at b, a the two points is negative (i.e., $f(\mathrm{b}), f(\mathrm{a})$ the function values at the points have opposite signs), then there must be at least one root within the interval $[a,b]$.



The two theorems mentioned above can be used to determine whether there is a root of the function $f(x)$ within a given interval $[a,b]$, according to the specified conditions.

Notice that if $f(\mathrm{a}), f(b) > 0$ (i.e., the values of the function at b, a both points have the same sign), this means either there is no root within the interval or there is an even number of roots within that interval $[a,b]$.

Example:

Determine the locations of the roots of the function $f(x)$ within the interval $[-3,3]$.

$$f(x) = x^3 - 6x + 2$$

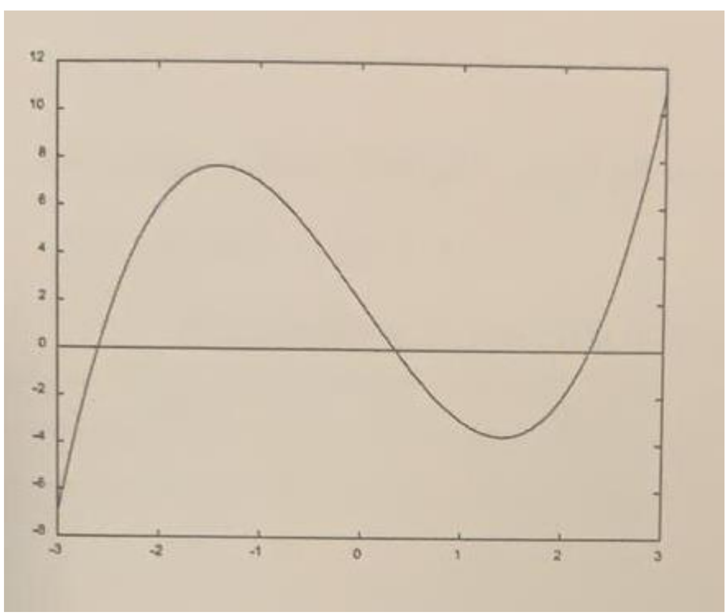
Solution:

Given that the interval is relatively wide, we divide the interval $[-3,3]$ into smaller subintervals with a fixed increment, and then we determine the sign of the function at the boundaries of each subinterval as follows:

$$[-3,3] \rightarrow [-3,-2],[-2,-1],[-1,0],[0,1],[1,2],[2,3]$$

x	-3	-2	-1	0	1	2	3
$f(x)$	-7	6	7	2	-3	-2	11

From the table, we observe that whenever the sign of the function changes from positive to negative or vice versa, it indicates the presence of roots within the subintervals. Thus, we conclude that there are roots within the intervals $[-3,2],[0,1],[2,3]$, because the sign of the function changes at the boundaries of these intervals.



Another Example:

Determine the locations of the roots of the function $f(x)$ within the interval $[3,10]$

$$f(x) = x^2 \sin(x) - \ln(x)$$

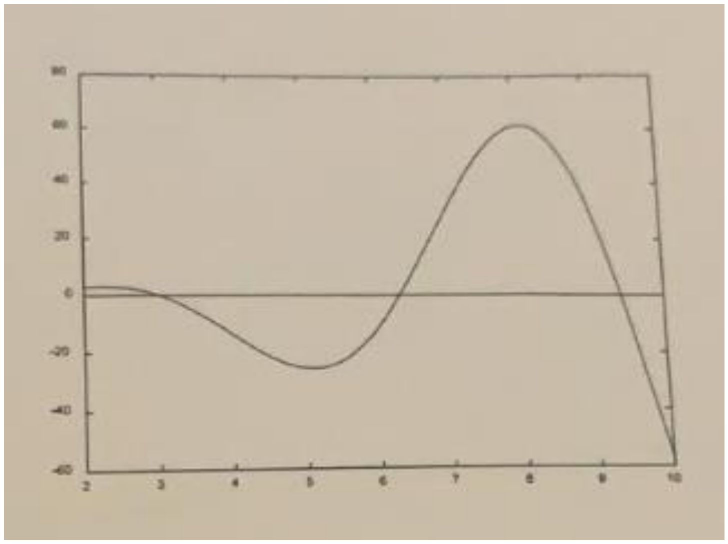
Solution:

We divide the interval as follows:

$$[3,10] \rightarrow [3,4],[4,5],[5,6],[6,7],[7,8],[8,9],[9,10]$$

x	3	4	5	6	7	8	9	10
$f(x)$	0.17	-13.50	-25.58	-11.85	30.25	61.24	31.18	-56.7

From the table, we conclude that there are roots within the intervals $[3,4],[6,7],[9,10]$, as the sign of the function changes at the boundaries of these intervals.



Exercise:

Apply the root location method to the example mentioned in Lecture 3, when $f(x) = 3x^3 - 4x + 1$ within the interval $[-1.5, 1.5]$, using an increment of 0.5.

x	-1.5	-1	-0.5	0	0.5	1	1.5
$y = f(x)$							