1- Using Gates to Represent Formulas

At the basic hardware level, computer memory has two states, which are identified as the two **logical values** or **Boolean values** of **T** and **F**. Computer operations are thought of as being composed of operations on these Boolean values and, hence, as operations of propositional logic. In describing computer circuits, a specialized notation for propositional logic is used. Special physical devices, called gates, implement the ∧, ∨, and ¬ operations. A set of gates to represent a circuit is called a **combinatorial circuit** or **combinatorial network**.

Think of a gate as representing an operation and of the wires going into the gates as representing its operands. For example, a A gate will let current flow out if and only if both operands (that is, both wires coming in) carry current. Notation for these gates is shown in Figure 2.1.

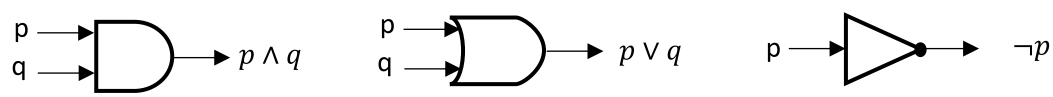

Figure 2.1. AND, OR, NOT gates

A combinatorial circuit is, roughly, the analogue of a formula. Boolean circuit notation for the formula.

$$((p \wedge q) \wedge r)$$

is shown in Figure 2.2.

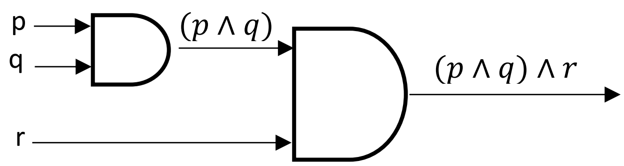

Figure 2.2. AND gates.

For the formula

$$((p \wedge p) \wedge p)$$

instead of having three separate p's as in an expression tree, the gate to represent it has one line that split, as shown in Figure 2.3.

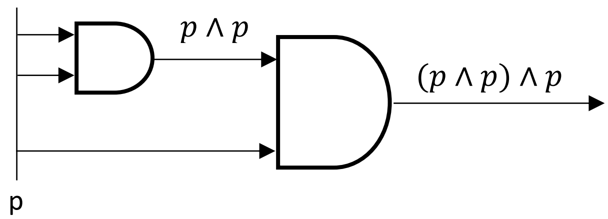

Figure 2.3. Another form of AND gates.

Table 1 lists many commonly used formulas. The reader should study them carefully.

Table 2.1: Commonly used formula			
1	$(p \wedge p) \leftrightarrow p$	Unchanged	
2	$(p \vee p) \leftrightarrow p$	Unchanged	
3	$((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$	Associative Law	القانون الترابطي
4	$((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$	Associative Law	
5	$((p \leftrightarrow q) \leftrightarrow r) \leftrightarrow (p \leftrightarrow (q \leftrightarrow r))$	Associative Law	
6	$(p \wedge r) \leftrightarrow (r \wedge p)$	Commutative Law	القانون تبادلي
7	$(p \vee r) \leftrightarrow (r \vee p)$	Commutative Law	
8	$(p \leftrightarrow r) \leftrightarrow (r \leftrightarrow p)$	Commutative Law	
9	$(p \wedge (r \vee q)) \leftrightarrow ((p \wedge r) \vee (p \wedge q))$	Distributive Law	قانون التوزيع
10	$(p \vee (r \wedge q)) \leftrightarrow ((p \vee r) \wedge (p \vee q))$	Distributive Law	
11	$\neg\neg p \leftrightarrow p$	Double negative	نفي مزدوج
12	$\neg(p \wedge r) \leftrightarrow (\neg p \vee \neg r)$	DeMorgan's Law	قانون دي مورغان
13	$\neg(p \vee r) \leftrightarrow (\neg p \wedge \neg r)$	DeMorgan's Law	
14	$(p \rightarrow r) \leftrightarrow (\neg r \rightarrow \neg p)$	Contrapositive	مانع للإيجابية
16	$((\neg p \rightarrow r) \wedge (\neg p \rightarrow \neg r)) \leftrightarrow p$	Contradiction	تناقض
17	$((p \wedge r) \vee r) \leftrightarrow r$	Absorption	
18	$((p \vee r) \wedge r) \leftrightarrow r$	Absorption	
19	$(p \rightarrow F) \leftrightarrow (\neg p)$		

Exercises

1. Translate the following expressions into propositional logic. Use the following proposition letters: **p** = "Jones told the truth." **q** = "The butler did it." **r** = "I'll eat my hat." **s** = "The moon is made of green cheese." **t** = "If water is heated to 100 0 C, it turns to vapor."

a. "If Jones told the truth, then if the butler did it, I'll eat my hat."
b. "If the butler did it, then either Jones told the truth or the moon is made of green cheese, but not both."
c. "It is not the case that both Jones told the truth, and the moon is made of green cheese."
d. "Jones did not tell the truth, and the moon is not made of green cheese, and I'll not eat my hat."
e. "If Jones told the truth implies, I'll eat my hat, then if the butler did it, the moon is made of green cheese."
f. "Jones told the truth, and if water is heated to 100 ^{o}C, it turns to vapor."

Solution.

1. (a) $p \rightarrow (q \rightarrow r)$

1. (b) $q \rightarrow ((p \lor s) \land \neg (p \land s))$

1. (c) $\neg (p \land s)$

1. (d) $\neg p \land \neg s \land \neg r$

1. (e) $(p \rightarrow r) \rightarrow (q \rightarrow s)$

1. (f) $p \land t$

2. Let ***p*** denote the proposition "Jill plays basketball" and ***q*** denote the proposition "Jim plays soccer." Write out-in the clearest way you can-what the following propositions mean:

(a) $\neg p$

(b) $p \land q$

(c) $p \lor q$

(d) $\neg p \land q$

(e) $p \rightarrow q$

(f) $p \leftrightarrow q$

(g) $\neg q \rightarrow p$

Solution.

2. (a) Jill does not play basketball.

2. (b) Jill plays basketball and Jim plays soccer.

2. (c) Jill plays basketball or Jim plays soccer.

2. (d) Jill does not play basketball and Jim plays soccer.

2. (e) If Jill plays basketball then Jim plays soccer.

2. (f) Jill plays basketball if and only if Jim plays soccer.

2. (g) If Jim does not play soccer then Jill plays basketball.

3. Let proposition ***p*** be **T**, proposition ***q*** be **F**, and proposition ***r*** be **T**. Find the truth values for the following:

(a) $p \lor q \lor r$

(b) $p \lor (\neg q \land \neg r)$

(c) $p \rightarrow (q \lor r)$

(d) $(q \land \neg p) \leftrightarrow r$

(e) $\neg r \rightarrow (p \land q)$

(f) $(p \rightarrow q) \rightarrow \neg r$

(g) $((p \land r) \rightarrow (\neg q \lor p)) \rightarrow (q \lor r)$

Solution.

3.(a) T

3.(b) T

3.(c) T

3.(d) F

3.(e) T

3.(f) T

3.(g) T