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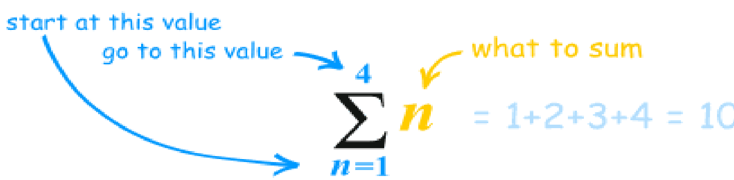
Summation notation and properties



Sigma is the upper case letter S in Greek. And S stands for Sum.

Example:

The sum $u_1 + u_2 + u_3 + ... + u_n$ is written in sigma notation as $\sum_{i=1}^{n} u_i$.

Example: Evaluate $\sum_{r=1}^{4} r^3$

Solution:

$$\sum_{r=1}^{4} r^3 = 1^3 + 2^3 + 3^3 + 4^3$$

$$= 1 + 8 = 27 + 64 = 100$$

Example: Evaluate $\sum_{n=2}^{5} n^2$

Solution:

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2$$

$$= 4 + 9 + 16 + 25 = 54$$

Example: Evaluate $\sum_{k=0}^{5} 2^k$

Solution:

$$\sum_{k=0}^{5} 2^k = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$
$$= 1 + 2 + 4 + 8 + 16 + 32$$
$$= 63$$

Example: Evaluate $\sum_{r=1}^{4}(-1)^r$

Solution: here, we need to remember that $(-1)^2 = +1, (-1)^3 = -1$, and so on, so

$$\sum_{r=1}^{4}(-1)^r = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4$$
$$= (-1) + 1 + (-1) + 1 = 0$$

Example: Evaluate $\sum_{k=1}^{3}\left(-\frac{1}{k}\right)^2$

Solution: once again, we must remebmer how to deal with powers of -1:

$$\sum_{k=1}^{3}\left(-\frac{1}{k}\right)^2 = \left(-\frac{1}{1}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{3}\right)^2$$
$$= 1 + \frac{1}{4} + \frac{1}{9} =$$

Example: write the sum

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots+\frac{1}{100}$$

In sigma notation.

Solution: we can now see that k-th term is $(-1)^k \, {}^{1}\!/_{k}$, and that there are 100 term, so we would write the sum in sigma notation as

$$\sum_{k=1}^{100}(-1)^k\frac{1}{k}$$

<u>Rules for use with sigma notation</u>

If a and c are constants, and if $f(k)$ and $g(k)$ are functions of k, then

(1) $\sum_{k=1}^{n} c = c + c + c + \cdots . + c = nc$

(2) $\sum_{k=1}^{n} ck = (c \times 1) + (c \times 2) + \cdots (c \times n)$

$$= c \times (1 + \cdots + n)$$

$$c\sum_{k=1}^{n} k$$

(3) $\sum_{k=1}^{n}(k + c) = (1 + c) + (2 + c) + \cdots + (n + c)$

$$(c + c + \cdots + c) + (1 + 2 + \cdots + n)$$

$$= nc + \sum_{k=1}^{n} k$$

(4) $\sum_{k=1}^{n}(ag(k) + c) = nc + a\sum_{k=1}^{n} g(k)$

(5) $\sum_{k=1}^{n}\big(f(k)+g(k)\big)=\sum_{k=1}^{n}f(k)+\sum_{k=1}^{n}g(k)$

Extra Examples:

(1) $\sum_{n=1}^{4}(2n+1)=3+5+7+9=24$

(2) $\sum_{i=1}^{3}i(i+1)=1\times 2+2\times 3+3\times 4=20$

(3) $\sum_{i=3}^{5}\frac{i}{i+1}=\frac{3}{4}+\frac{4}{5}+\frac{5}{6}$

(4) $(x_1-\mu)^2+(x_2-\mu)^2+(x_3-\mu)^2+(x_4-\mu)^2=\sum_{k=1}^{4}(x_k-\mu)^2$

Note:

If $f(i)$ represents some expression (function) involving i, then

$$\sum_{i=1}^{n}f(i)=f(1)+f(2)+f(3)+\cdots+f(n)$$

Example:

$$\sum_{i=1}^{4}(2+i^2)=(2+1^2)+(2+2^2)+(2+3^2)+(2+4^2)$$

$$=3+6+11=18=38$$

The well-known summation rules:

1. $\sum_{i=1}^{n}c=c+c=c=\cdots+c$ (n times)=nc, where c is a constant
2. $\sum_{i=1}^{n}i=1+2+3+\cdots+n=\frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n}i^2=1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

Arithmetic and geometric series

two common types of mathematical sequences are arithmetic sequences and geometric sequences. An arithmetic sequence has a constant difference between each consecutive pair of terms. This is similar to the linear functions that have the form $y = mx + b$. A geometric sequence has a constant ratio between each pair of consecutive terms. This would create the effect of a constant multiplier.

Examples

Arithmetic Sequence:

{5,11,17,23,29,35,…}

Notice here the constant difference is 6. If we wanted to write a general term for this sequence, there are several approaches. One approach is to take the constant difference as the coefficient for the n term: $a_n = 6n+?$ Then we just need to fill in the question mark with a value that matches the sequence. We could say for the sequence:

{5,11,17,23,29,35,…}

$a_n = 6n - 1$

There is also a formula which you can memorize that says that any arithmetic sequence with a constant difference d is expressed as:

$$a_n = a_1 + (n-1)d$$

Notice that if we plug in the values from our example, we get the same answer as before:

$$a_n = a_1 + (n-1)d$$

$$a_1 = 5, d = 6$$

So, $a_1 + (n-1)d = 5 + (n-1) \times 6 = 5 + 6n - 6 = 6n - 1$

If the terms of an arithmetic sequence are getting smaller, then the constant difference is a negative number.

{24,19,14,9,4,–1,–6,…}

$$a_n = -5n + 29$$

Geometric Sequence

In a geometric sequence there is always a constant multiplier. If the multiplier is greater than 1, then the terms will get larger. If the multiplier is less than 1, then the terms will get smaller.

$$\{2,6,18,54,162, \dots\}$$

Notice in this sequence that there is a constant multiplier of 3. This means that 3 should be raised to the power of n in the general expression for the sequence. The fact that these are not multiples of 3 means that we must have a coefficient before the 3^n

$$\{2,6,18,54,162, \dots\}$$

$$a_n = 2 * 3^{n-1}$$

If the terms are getting smaller, then the multiplier would be in the denominator:

{50,10,2,0.4,0.08,…}

Notice here that each term is begin divided by 5 (or multiplied by $\frac{1}{5}$).

{50,10,2,0.4,0.08,….}

$$a_n = \frac{50}{5^{n-1}} \text{ or } a_n = \frac{250}{5^n} \text{ or } a_n = 50 * \left(\frac{1}{5}\right)^{n-1}$$

Exercises:

Determine whether each sequence is arithmetic, geometric or neither. If it is arithmetic, determine the constant difference. If it is geometric determine the constant ratio.

1) $\{18,22,26,30,34, \dots\}$
2) $\{9,19,199,1999, \dots\}$
3) $\{8,12,18,27, \dots\}$
4) $\{15,7,-1,-9,-17, \dots\}$
5) $\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6}, \dots\right\}$
6) {100,−50,25,−12.5,…}