The Discrete Fourier Transform

In the two-variable case the discrete Fourier transform pair is given by the equations:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M + vy/N)]$$

for u=0,1,2, . . . , M-1   , v=0,1,2, . . . ,N-1

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M + vy/N)]$$

for x=0,1,2 , . . . , M-1 and   y=0,1,2 , . . . , N-1

sampling of a continuous function is now in a two-dimensional grid with divisions of width Δx and Δy in the x- and y- axis respectively. As in the one-dimensional case, the discrete function $f(x,y)$ represents samples of the function $f(x_0 + x\Delta x, y_0 + y\Delta y)$ for *x=0,1,2,…,M-1* and y*=0,1,2,…,N-1* similar comments hold for *F(u,v)*. The sampling increments in the spatial and frequency domains are related by:

$\Delta u = \dfrac{1}{M\Delta x}$ and $\Delta v = \dfrac{1}{N\Delta y}$

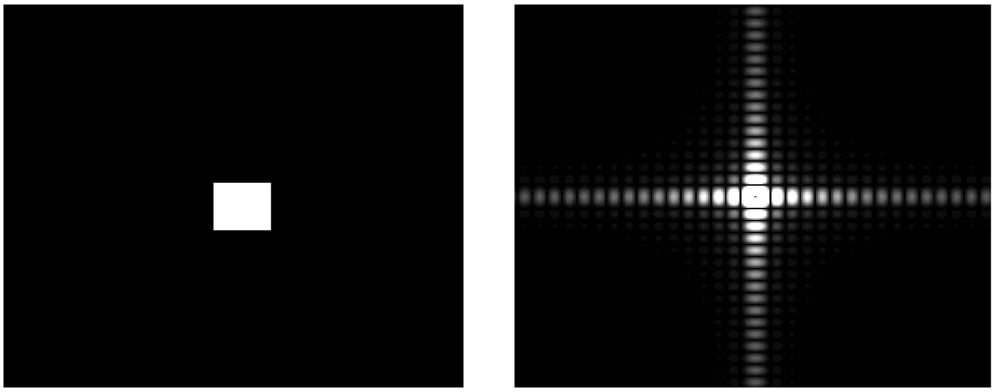
when images are sampled in a square array we have that *M=N* and

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux + vy)/N]$$

for u,v=0,1,2, . . . ,N-1

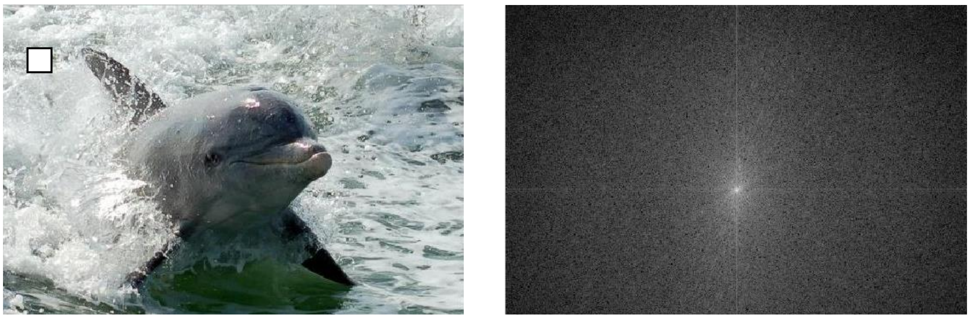
$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux + vy)/N]$$

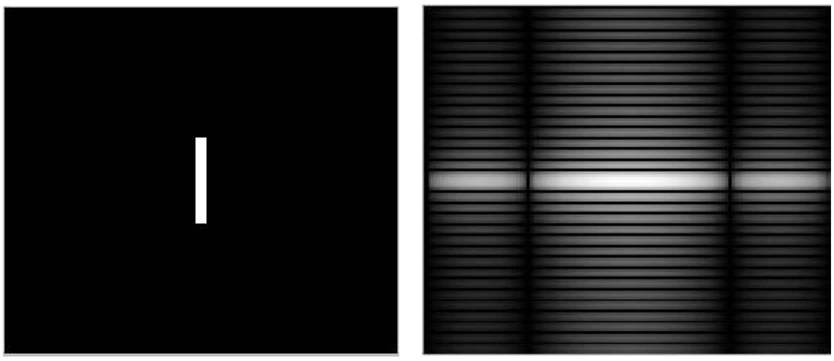
for x,y=0,1,2 , . . . , N-1

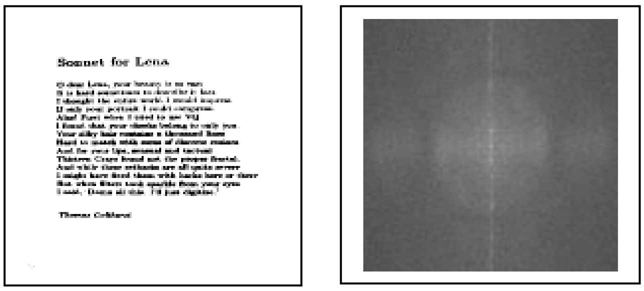


(a) *(b)*

Figure: (a) square image , (b) DFT magnitude of the square image







Example: ***Find the DFT of the following function***

$$f(x,y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}_{4*4}$$

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux+vy)/N]$$

$$\begin{aligned} F(0,0) = \frac{1}{4}[(&f(0,0)\ e^{-j2\pi(u*0+v*0)/N} + f(0,1)\ e^{-j2\pi(u*0+v*1)/N} + \\ &f(0,2)\ e^{-j2\pi(u*0+v*2)/N} + f(0,3)\ e^{-j2\pi(u*0+v*3)/N}) \\ &+ (f(1,0)\ e^{-j2\pi(u*1+v*0)/N} + f(1,1)\ e^{-j2\pi(u*1+v*1)/N} + \\ &f(1,2)\ e^{-j2\pi(u*1+v*2)/N} + f(1,3)\ e^{-j2\pi(u*1+v*3)/N}) \\ &+ (f(2,0)\ e^{-j2\pi(u*2+v*0)/N} + f(2,1)\ e^{-j2\pi(u*2+v*1)/N} + \\ &f(2,2)\ e^{-j2\pi(u*2+v*2)/N} + f(2,3)\ e^{-j2\pi(u*2+v*3)/N}) \\ &+ (f(3,0)\ e^{-j2\pi(u*3+v*0)/N} + f(3,1)\ e^{-j2\pi(u*3+v*1)/N} + \\ &f(3,2)\ e^{-j2\pi(u*3+v*2)/N} + f(3,3)\ e^{-j2\pi(u*3+v*3)/N})] \end{aligned}$$

$$\begin{aligned} = \frac{1}{4}[\ &f(0,0) + f(0,1) + f(0,2) + f(0,3) \\ &+ f(1,0) + f(1,1) + f(1,2) + f(1,3) \\ &+ f(2,0) + f(2,1) + f(2,2) + f(2,3) \\ &+ f(3,0) + f(3,1) + f(3,2) + f(3,3)] \end{aligned}$$

$$= \frac{1}{4}\ [0 + 0 + 1 + 0 + 2 + 0 + 0 + 0 + 0 + 3 + 0 + 0 + 0 + 1 + 0 + 2]$$

$$= \frac{1}{4}\ [9]$$

$$= 2.25$$

properties of the two-dimensional Fourier transform

1. *Separability*

The separability property of a two dimensional transform and its inverse ensures that such computations can be performed by decomposing the two dimensional transforms into two one dimensional transforms. From Equations of *DFT* and inverse *DFT* of a two dimensional function *f(x,y)*, we can express them in separable form as follows:

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v)\exp[-j2\pi\, ux/N]$$

Where

$$F(x,v) = N\left[\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi\ \mathrm{vy}/N]\right]$$

Hence the two dimensional *DFT* (as well as the inverse *DFT*) can be computed by the taking the one dimensional *DFT* row-wise in the two dimensional image and the result is a gain transformed column-wise by the same one dimensional *DFT*. As shown in the following figure.

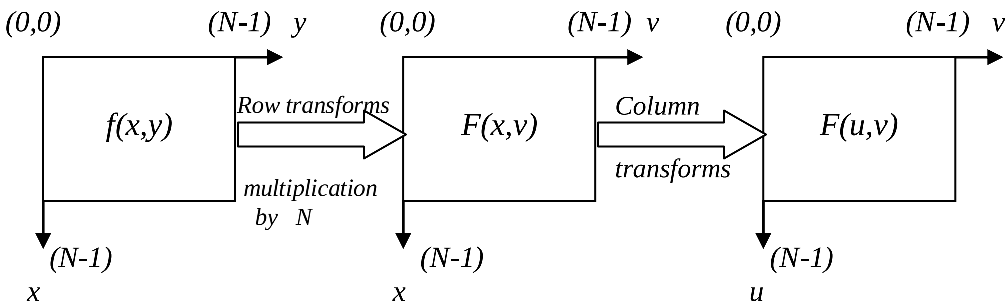


Fig. : Computation of the two dimensional Fourier transform as a series of one dimensional transforms

2. *Translation*

The translation properties of the fourier transform pair are given by:

$$f(x,y)\exp[j2\pi(u_0x+v_0y)/N] \Leftrightarrow F(u-u_0, v-v_0)$$

and

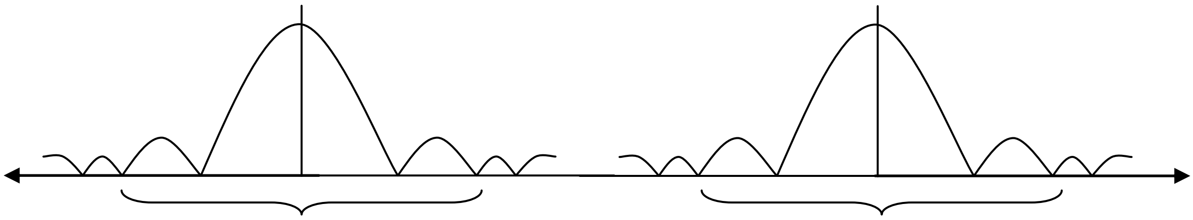
$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\exp[-j2\pi(ux_0+vy_0)/N]$$

Where the double arrow is used to indicate the correspondence between a function and its fourier transform.

3. *Periodicity*

The *DFT* of a two dimensional function *f(x,y)* and its inverse are both periodic with period τ , i.e,

$F(u,v)=F(u+\tau,v)=F(u,v+\tau)=F(u+\tau,v+\tau)$.

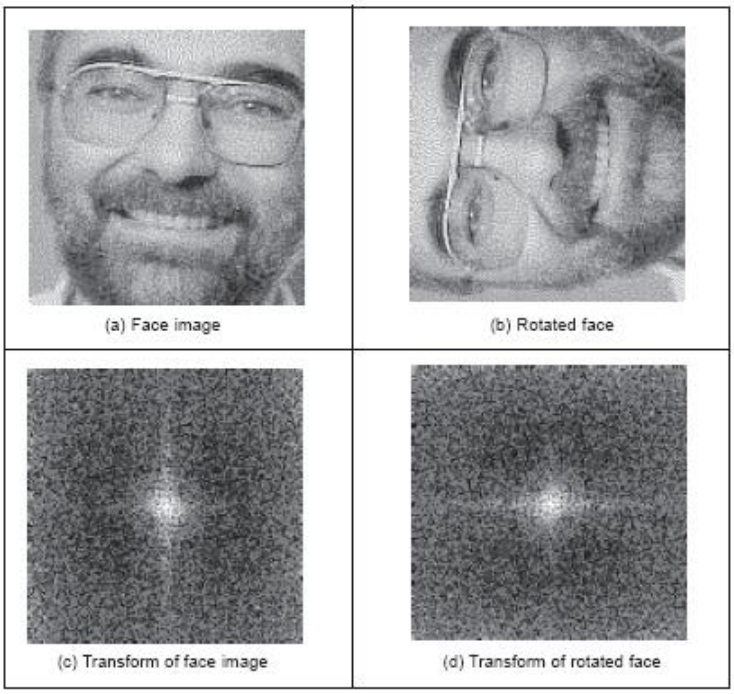


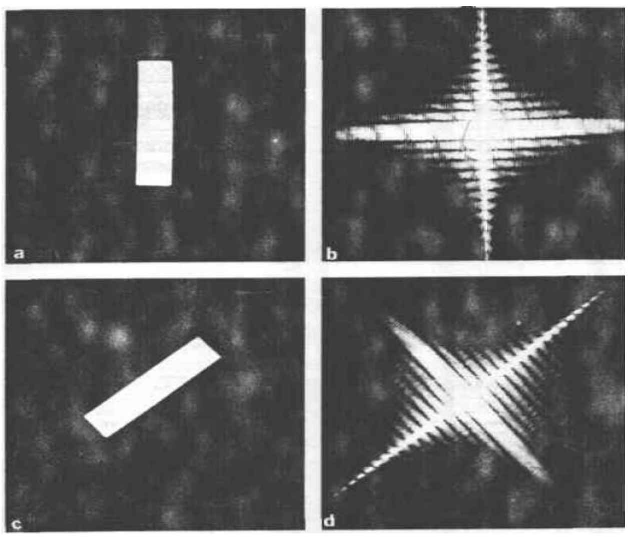
τ *One period* τ *One period*

4. *Rotation*

Assuming that the function *f(x,y)* undergoes a rotation of α , the corresponding function *f(x,y)* in polar coordinates will then be represented as $f(r,\alpha)$, where $x=rcose\alpha$ and $y=rsin\alpha$. The corresponding *DFT F(u,v)* in polar coordinates will be represented as $F(\beta,\gamma)$, where $u=\beta\cos\gamma$ and $v=\beta\sin\gamma$.

The above implies that if *f(x,y)* is rotated by α_0, then *F(u,v)* will be rotated by the same angle α_0 and hence we can imply that $f(r,\alpha+\alpha_0)$ corresponds to $F(\beta,\gamma+\alpha_0)$ in the *DFT* domain and vice versa.





Rotational properties of DFT (a) a simple image. (b) Spectrum. (c) Rotated image. (d) Resulting spectrum

5. *Distributive Property*

The *DFT* of sum of two functions $f_1(x,y)$ and $f_2(x,y)$ is identical to the sum of the *DFT* of these two functions, *i.e.*,

$\Im\{f_1(x,y)+f_2(x,y)\}= \Im\{f_1(x,y)\}+ \Im\{ f_2(x,y)\}$

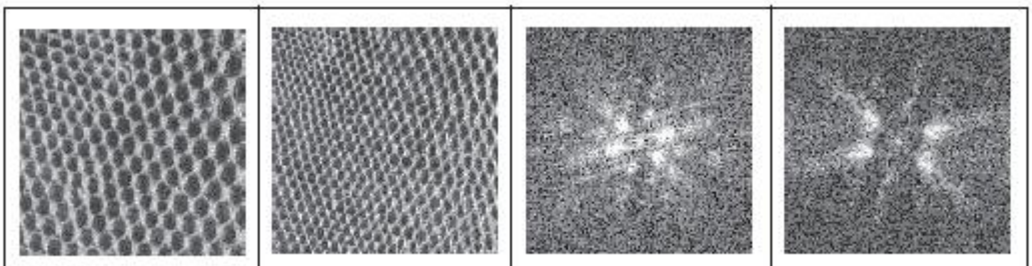
where $\Im\{f_1(x,y)\}$ is the *DFT* of $f_1(x,y)$. It should be noted that the distributive property for product of two functions does not hold *i.e*,

$\Im\{f_1(x,y).\, f_2(x,y)\} \neq \Im\{f_1(x,y)\}\, .\, \Im\{ f_2(x,y)\}$

6. *Scaling*

The *DFT* of a function $f(x,y)$ multiplied by a scalar (k) is identical to the multiplication of the scalar with the *DFT* of the function $f(x,y)$, *i.e*, $\Im\{ k\, f(x,y)\}= k\, F(u,v)$.

$$\Im\{f(ax,by)\}= \frac{1}{|ab|} F\left(\frac{u}{a},\frac{v}{b}\right)$$



(a) Texture image (b) Scaled texture image (c) Transform of original texture (d) Transform of scaled texture

7. *Average Value*

A widely used definition of the average value of a two dimensional discrete function is given by the expression:

$$\bar{f}(x,y) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

substitution of $u=v=0$ in Eq. of *DFT* yields:

$$F(0,0) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

we see, therefore, that $\bar{f}(x,y)$ is related to the fourier transform of $f(x,y)$ by the equation:

$$\bar{f}(x,y) = \frac{1}{N} F(0,0)$$