Properties of the two-dimensional Fourier transform

8. Convolution

The *DFT* of convolution of two functions is equal to the product of the DFT of these two functions, i.e.

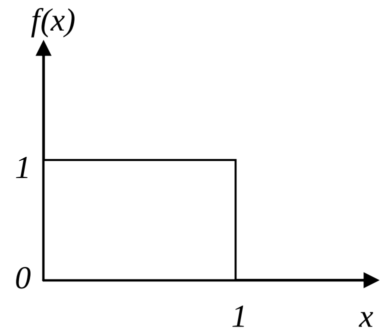
$\Im\{ f_1(x,y) * f_2(x,y) \} = F_1(u,v) \,.\, F_2(u,v)$

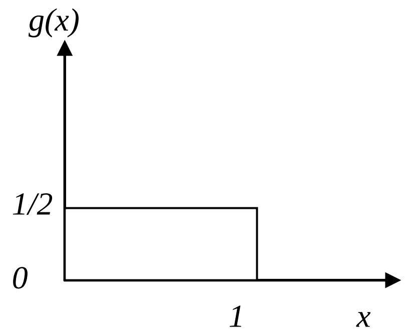
The convolution of two functions *f(x)* and *g(x)* denoted by *f(x)*g(x)* is defined by the integral:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

Where α is a dummy variable of integration. The convolution of two functions *f(x)* and *g(x)* before carrying out the integration it is necessary to form the function $g(x-\alpha)$. It is noted that this operation is simply one of folding $g(\alpha)$ about the origin to give $g(-\alpha)$ and then displacing this function by *x*. Then for any given value of *x*, we multiply $f(\alpha)$ by the corresponding $g(x-\alpha)$ and integrate the product from $-\infty$ to ∞.

<u>Example</u> : convolved the following functions:





<u>Solution:</u>

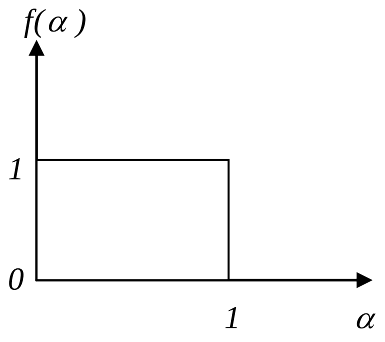
$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \qquad g(x) = \begin{cases} 1/2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

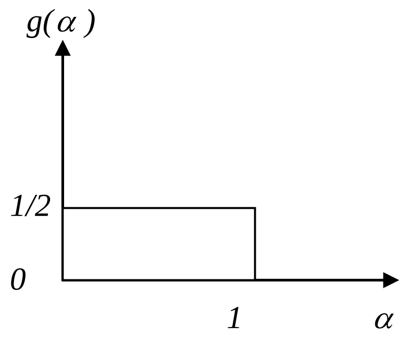
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha) g(x-\alpha) d\alpha$$

to convolved between two functions we must followed the following steps:

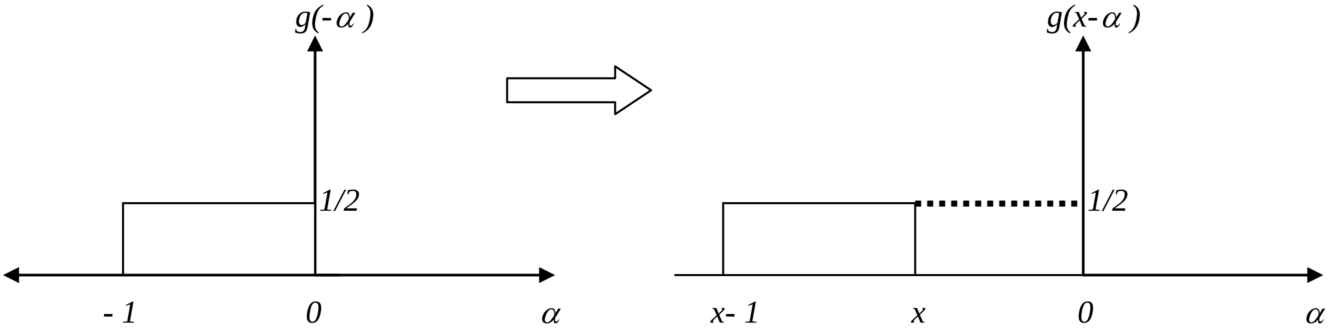
- ***graphically***

step1: change the axis





Step2: folding g(α) about origin to give g(-α) and then displacing this function by x to become g(x-α)



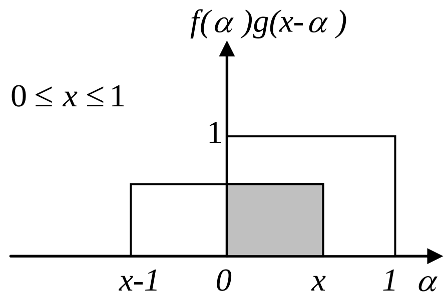
Step3: Sliding , for any given value of x , we multiply f(α) by the corresponding g(x-α) and integrate the product from -∞ to ∞.

(1) when x<0 $\quad f(x)*g(x)=0$

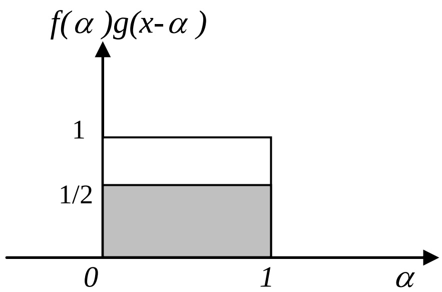
(2) when $0 \le x \le 1$

$$\int_0^x f(\alpha)g(x-\alpha)d\alpha$$

$$= \int_0^x 1.\frac{1}{2}d\alpha = \frac{1}{2}\alpha\Big|_0^x = \frac{1}{2}\,x$$



(3) this interval satisfied by step (2) and (4)

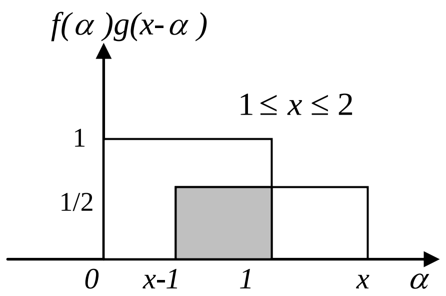


(4) when $1 \ge x-1 \ge 0$

$$2 \ge x \ge 1$$

$$\int_{x-1}^{1} \frac{1}{2}.1d\alpha$$

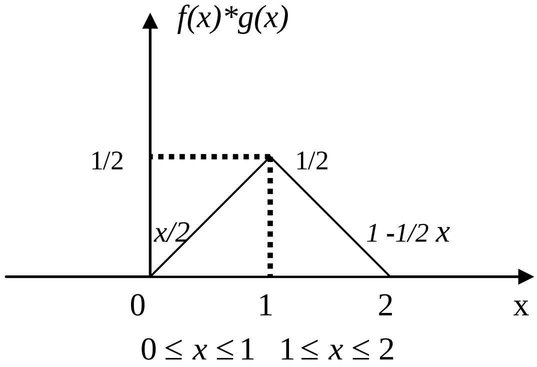
$$= \frac{1}{2}\alpha\Big]_{x-1}^{1} = \frac{1}{2} - \frac{1}{2}(x-1) = 1 - \frac{1}{2}x$$



(5) when $x-1>1 \quad \Rightarrow \quad x>2$

*f(x)*g(x) = 0*

$$f(x)*g(x)=\begin{cases}1/2x & 0\le x\le 1\\ 1-\frac{1}{2}x & 1\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$



- ***mathematically***

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)d\alpha \qquad 0\le\alpha\le 1 \quad \textit{for}\ f(\alpha)$$

$$0\le\alpha\le 1 \quad \textit{for}\ g(\alpha)$$

$$0\le\alpha\le 1$$
$$x-1\le\alpha\le x$$

$$\int_0^x f(\alpha)g(x-\alpha)d\alpha=\int_0^x 1.\frac{1}{2}d\alpha = \frac{1}{2}\alpha\Big|_0^x = \frac{1}{2}x$$

$$\int_{x-1}^{1}\frac{1}{2}.1d\alpha=\frac{1}{2}\alpha\Big]_{x-1}^{1} = \frac{1}{2}-\frac{1}{2}(x-1) = 1-\frac{1}{2}x$$

$$f(x)*g(x)=\begin{cases}1/2x & 0\le x\le 1\\ 1-\frac{1}{2}x & 1\le x\le 2\\ 0 & \text{otherwise}\end{cases}$$