Lecture 5: Numerical Methods for Solving Nonlinear Equations – Bisection Method

To determine the roots of nonlinear equations with the desired accuracy, we rely on numerical methods (also called iterative methods). These methods often lead to highly accurate estimates of the roots, especially when using computers to execute the iterative algorithm. In numerical methods, we typically need to assume one or more initial approximate values for the root, which serve as the initial guess. These initial values are used to run the iterative algorithm until the desired solution is reached with the specified accuracy. To assess the accuracy of the solution, it is essential to define the permissible absolute error, denoted by ε (epsilon), where the smaller the ε, the higher the solution's accuracy (see Lecture 1).

The Basic Idea of Iterative Methods:

1. **Determining the initial value of the root** x_0: It is preferable for these initial values to be close to the actual root to reduce the number of required iterations. Graphing the functions can help in selecting initial values.
2. **Defining the maximum allowable absolute error**: For example, $\varepsilon = 0.0001 = 1 \times 10^{-4}$

3-Applying the iterative algorithm: The algorithm is run once using the initial value x_0 to obtain a new estimate of the root x_1

4- Calculating the absolute error of the estimate, which is done as follows:

$\Delta x = \left| x_1 - x_0 \right|$ Then, the absolute error is compared to the value ε. If the condition is satisfied:

$\left|x_1 - x_0\right| \leq \varepsilon$ then we stop the algorithm and conclude that the approximate root is x_1 However, if the condition above fails (i.e., if $\left|x_1 - x_0\right| > \varepsilon$), this means we need to perform another iteration, returning to step 3, and using x_1 as the new initial value.to obtain x_2 The absolute error is then recalculated using:

$\left|x_2 - x_1\right|$and compared to the allowable tolerance ε. This process continues until the condition is satisfied:

$\left|x_n - x_{n-1}\right| \leq \varepsilon$ At this point, $n = 1, 2, 3$....... the value x_n represents the final approximate value of the root of the function.

Note: Since the purpose of the iterative approximation process is to obtain an approximate value for the root of the function $f(x)$, and since the function's value at the true root equals zero, the test $\left|f(x_n)\right| \leq \varepsilon$ can be used instead of $\left|x_n - x_{n-1}\right| \leq \varepsilon$, depending on the user's preference. The function's f value at the estimated root x_n should be very small (close to zero) if x_n is a good approximation of the true root. Therefore, the function value at x_n can be compared to a very small value, such as ε (epsilon), to decide whether to continue with further iterations or to stop the algorithm.

There are several iterative methods for solving nonlinear equations. In the following sections, we will review some of these methods, noting that the primary differences between them lie in the speed of convergence to the final solution, the number of steps required to reach the solution, and the accuracy of the results.

Bisection Method:

This method requires determining two initial values for the root x_a, x_b under the condition that $x_a < x_b$ and $f\left(x_a\right), f\left(x_b\right) < 0$ have opposite signs, meaning there must be a root within the interval $\left[x_a, x_b\right]$. Additionally, the function must be continuous on the interva$\left[x_a, x_b\right]$.

Note that if $f\left(x_a\right), f\left(x_b\right) = 0$, it means either x_a is the root, or x_b is the root, or both are roots of the equation.

The best and easiest way to choose the values x_a, x_b is by graphing the function and approximately locating the root (which, as mentioned earlier, is where the curve intersects the x-axis). Then, select two points that lie on either side of the root.

Next, calculate the midpoint $x_m = \dfrac{x_a + x_b}{2}$, which is considered the x_m proposed value for the root according to the Bisection Method. Here, we test whether the absolute value of the function at $\left|f(x_m)\right| \le \varepsilon$ is less than or equal to the allowable error level ε (epsilon). This is to determine whether x_m is the required root. If the condition is satisfied, we stop and declare that the value x_m is the desired approximation of the root.

Otherwise, if the absolute value of the function at x_m is greater than ε (epsilon) (i.e., $\left|f(x_m)\right| > \varepsilon$), we then define a new interval by applying the test:

- If $f(x_m), f(x_b) < 0$, it means the root lies within the interval $[x_m, x_b]$, so we set $x_a = x_m$.
- Conversely, if $f(x_m), f(x_b) > 0$, it means the root lies within the interval $[x_a, x_b]$, so we set $x_b = x_m$.

Thus, we will have a new interval $[x_a, x_b]$ whose length is half of the original interval (which is why this algorithm is called the Bisection Method). Note that the calculations will be more complex in the case of having more than one root within the interval $[x_a, x_b]$.

Summary of the Bisection Method Steps:

1. Determine the allowable error level, for example, $\varepsilon = 1\times 10^{-4}$.
2. Choose the initial values x_a, x_b such that $x_a < x_b$ and $f(x_a), f(x_b) < 0$.
3. Calculate the midpoint $x_m = \dfrac{x_a + x_b}{2}$, then compute the function value at x_m, compute $f(x_m)$
4. Test whether $|f(x_m)| \le \varepsilon$. If so, the root of the equation is x_m, and then we stop.

5-If $|f(x_m)| > \varepsilon$, ,we check whether $f(x_m), f(x_b) < 0$, and set $x_a = x_m$. If $f(x_m), f(x_b) > 0$, we set $x_b = x_m$. In this way, we obtain a new interval $[x_a, x_b]$. Then, we return to step 3 and apply the algorithm again to the new $[x_a, x_b]$ interval.

Example: Find the value of x that satisfies $f(x) = 3x^3 - 24 = 0$ using the bisection method. Use $\varepsilon = 0.0001$.

Note: We will round our calculations to four decimal places since $\varepsilon = 0.0001 = 1 \times 10^{-4}$

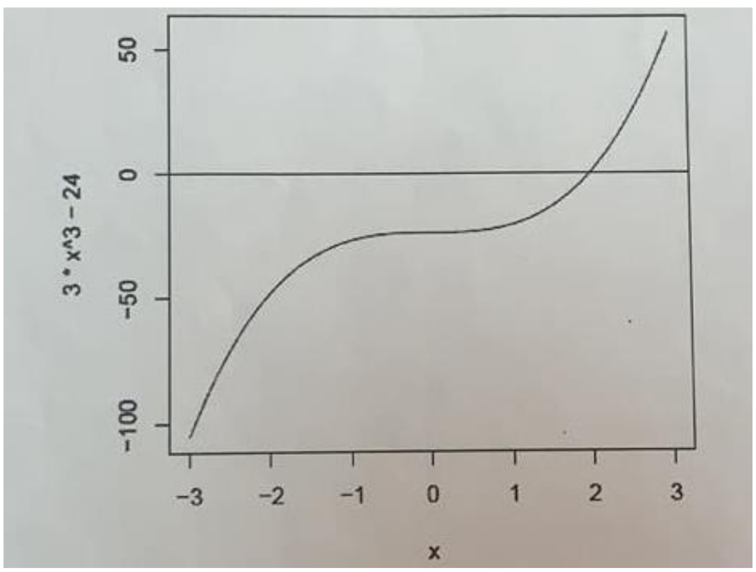
Solution:

Step 1: The value of ε is given in the question as $\varepsilon = 0.0001$.

Step 2: From the graph of the function, we can say that $[x_a, x_b] = [0, 3]$ are suitable initial values for the root of the function. Notice that the product of the function values at the endpoints is negative, i.e.,

$$f(x_a) = f(0) = 3 * 0^3 - 24 = -24$$
$$f(x_b) = f(3) = 3 * 3^3 - 24 = 57$$
$$f(x_a)\, f(x_b) = -24 * 57 < 0$$

This implies that the interval being tested contains a root.



Step 3: Calculate as $f\left(x_m\right), x_m$ follows:

$$x_m = \frac{x_a + x_b}{2} = \frac{0+3}{2} = 1.5$$

$$f\left(x_m\right) = f\left(1.5\right) = 3 * 1.5^3 - 24 = -13.875$$

Step 4: Compare the absolute value of the function at x_m with the value of ε as follows:

$$\left| f(x_m) \right| = 13.875 > \varepsilon$$

Since the value of the function is greater than the allowed error level $x_m = 1.5$ is not the required root.

Step 5: Find a new interval (choose one half of the previous interval). Notice that:

$$f(x_m).f(x_b) = -13.875 * 57 < 0$$

Since the product is negative, this means that the root is within the interval $\left[x_m, x_b\right] = \left[1.5, 3\right]$. Therefore, we set $x_a = x_m = 1.5$, and we have a new interval $\left[x_a, x_b\right] = \left[1.5, 3\right]$.

We then reapply **Step 3** to the new interval as follows:

$$x_m = \frac{x_a + x_b}{2} = \frac{1.5+3}{2} = 2.25$$

$$f\left(x_m\right) = f\left(2.25\right) = 3 * 2.25^3 - 24 = 10.1719$$

Step 4: Compare the absolute value of the function at x_m with ε as follows:

$$\left|f(x_m)\right| > \varepsilon$$

Therefore, $x_m = 2.25$ is not the required root.

Step 5: Find a new interval (choose one half of the previous interval). Notice that:

$$f(x_m).f(x_b) = 10.1719 * 57 > 0$$

Since the product is positive, this means that there is no root in the interval $\left[x_m, x_b\right] = \left[2.25, 3\right]$. Therefore, the root lies in the other half of the interval $\left[x_a, x_m\right] = \left[1.5, 2.25\right]$, and we set $x_b = x_m$, yielding a new interval, $\left[x_a, x_b\right] = \left[1.5, 2.25\right]$

We continue this process until we obtain the value of x_m that satisfies $\left|f(x_m)\right| \le \varepsilon$. In this example, the desired value of $x_m = 2$ is approximately. It takes 15 iterations to reach the required solution with the specified accuracy (rounded to four decimal places).

Example:

Find the root of $f\left(x\right) = x\ln x - 0.75 = 0$ using the bisection method, with $\varepsilon = 1\times10^{-4}$.

Note: The term $\ln\left(x\right)$ refers to the natural logarithm.

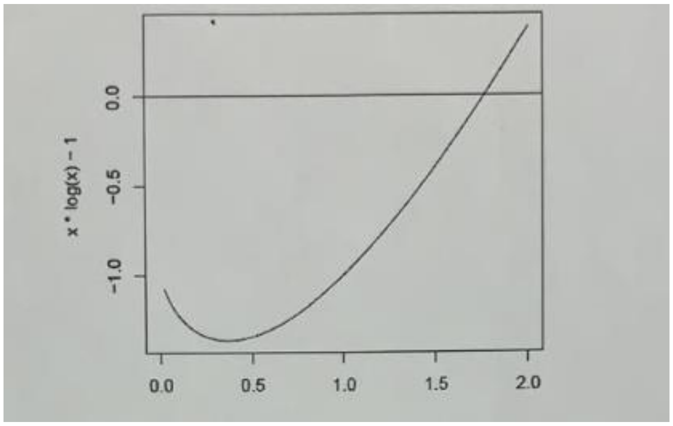
Solution:

From the graph of the function, we can say that the interval $[1,2]=[x_a, x_b]$ contains two suitable initial values for the root of the function. Note that the product of the function values is negative, meaning that:

$$f(x_a) = f(1) = 1*\ln(1) - 0.75 = -0.75$$

$$f(x_b) = f(2) = 2*\ln(2) - 0.75 = -0.6363$$

$$f(x_a)\, f(x_b) = -0.75*0.6363 < 0$$



To find the value of

$$x_m = \frac{x_a + x_b}{2} = \frac{0+3}{2} = 1.5$$

$$f(x_m = 1.5) = 1.5*\ln 1.5 - 0.75 = -0.1418$$

$$|f(x_m)| > \varepsilon$$

Therefore, $x_m = 1.5$ is not the required root.

Note that:

$$f\left(x_m\right) f\left(x_b\right) = -0.1418 * 0.6363 < 0$$

Thus, we set $x_a = x_m$, resulting in a new interval $\left[1,2\right] = \left[x_a, x_b\right]$We will reapply the algorithm to the new interval until we obtain a value x_m that satisfies $\left|f\left(x_m\right)\right| \leq \varepsilon$ In this example, the required iterations to reach the solution are 10 iterations.

The desired value is $x_m = 1.5987$.