Properties of the two-dimensional Fourier transform

8. Convolution

***<u>The convolution of</u>* <u>discrete function</u>**

Suppose that instead of being continuous f(x) and g(x) are discretized into sampled arrays of size A and B, respectively:
$\{f(0), f(1), f(2), \ldots\ldots\ldots, f(A-1)\}$ *, and*
$\{g(0), \mathrm{g}(1), \mathrm{g}(2), \ldots\ldots\ldots, g(B-1)\}$.

The discrete Fourier transform and its inverse are periodic functions. In order to formulate a discrete convolution theorem that is consistent with this periodicity property , we may assume that the discrete functions *f(x)* and *g(x)* are periodic with some period M. the resulting convolution will then be periodic with the same period. The value of *M* is selected as: *M=A+B-1.*

Since the assumed period must be greater than either *A* or *B*. the length of the sampled sequences must be increased so that both are of length *M.* this can be done by appending zeros to the given samples to form the following extended sequences:

$$f_e(x) = \begin{cases} f(x) & 0 \le \mathrm{x} \le \mathrm{A-1} \\ 0 & \mathrm{A} \le \mathrm{x} \le \mathrm{M-1} \end{cases}$$

and

$$g_e(x) = \begin{cases} g(x) & 0 \le \mathrm{x} \le B\mathrm{-1} \\ 0 & \mathrm{B} \le \mathrm{x} \le \mathrm{M-1} \end{cases}$$

Based on this we define the discrete convolution of $f_e(x)$ *and* $g_e(x)$ by the expression:

$$f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

for x=0,1,2, …. , M-1. The convolution function is a discrete, periodic array of length *M* , with the values *x=0,1,2 , … , M-1* describing a full period of $f_e(x)*g_e(x)$.

Example: Convolved the following functions

f = { 3 , 2.5 , 1 , 0.5 }
g = { 2 , 5 , 7 , 9 }
f contains 4 elements
g contains 4 elements
M = A+ B – 1 = 4 + 4 – 1 = 7 the interval 0 . . . 6

$$h_e(x) = f_e(x) * g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x-m)$$

$$
\begin{aligned}
h_e(0) &= f_e(0).g_e(0) + f_e(1).g_e(-1) + f_e(2).g_e(-2) + f_e(3).g_e(-3) \\
&\quad + f_e(4).g_e(-4) + f_e(5).g_e(-5) + f_e(6).g_e(-6) \\
&= 3 * 2 = \mathbf{6}
\end{aligned}
$$

$$
\begin{aligned}
h_e(1) &= f_e(0).g_e(1) + f_e(1).g_e(0) + f_e(2).g_e(-1) + f_e(3).g_e(-2) \\
&\quad + f_e(4).g_e(-3) + f_e(5).g_e(-4) + f_e(6).g_e(-5) \\
&= 3 * 5 + 2.5 * 2 = 15 + 5 = \mathbf{20}
\end{aligned}
$$

$$
\begin{aligned}
h_e(2) &= f_e(0).g_e(2) + f_e(1).g_e(1) + f_e(2).g_e(0) + f_e(3).g_e(-1) \\
&\quad + f_e(4).g_e(-2) + f_e(5).g_e(-3) + f_e(6).g_e(-4) \\
&= 3 * 7 + 2.5 * 5 + 1 * 2 = 21 + 12.5 + 2 = \mathbf{35.5}
\end{aligned}
$$

$$
\begin{aligned}
h_e(3) &= f_e(0).g_e(3) + f_e(1).g_e(2) + f_e(2).g_e(1) + f_e(3).g_e(0) \\
&\quad + f_e(4).g_e(-1) + f_e(5).g_e(-2) + f_e(6).g_e(-3) \\
&= 3 * 9 + 2.5 * 7 + 1 * 5 + 0.5 * 2 = 27 + 17.5 + 5 + 1 = \mathbf{50.5}
\end{aligned}
$$

$$
\begin{aligned}
h_e(4) &= f_e(0).g_e(4) + f_e(1).g_e(3) + f_e(2).g_e(2) + f_e(3).g_e(1) \\
&\quad + f_e(4).g_e(0) + f_e(5).g_e(-1) + f_e(6).g_e(-2) \\
&= 3 * 0 + 2.5 * 9 + 1 * 7 + 0.5 * 5 + 0 * 2 = \mathbf{32}
\end{aligned}
$$

$$
\begin{aligned}
h_e(5) &= f_e(0).g_e(5) + f_e(1).g_e(4) + f_e(2).g_e(3) + f_e(3).g_e(2) \\
&\quad + f_e(4).g_e(1) + f_e(5).g_e(0) + f_e(6).g_e(-1) \\
&= 3 * 0 + 2.5 * 0 + 1 * 9 + 0.5 * 7 = 9 + 3.5 = \mathbf{12.5}
\end{aligned}
$$

$$
\begin{aligned}
h_e(6) &= f_e(0).g_e(6) + f_e(1).g_e(5) + f_e(2).g_e(4) + f_e(3).g_e(3) \\
&\quad + f_e(4).g_e(2) + f_e(5).g_e(1) + f_e(6).g_e(0) \\
&= 0.5 * 9 = \mathbf{4.5}
\end{aligned}
$$

- by matrix

*Matrix = M*M*

$$\begin{pmatrix} g_e(0) & g_e(-1) & g_e(-2) & \cdots & g_e(-M+1) \\ g_e(1) & g_e(0) & g_e(-1) & \cdots & g_e(-M+2) \\ g_e(2) & g_e(1) & g_e(0) & \cdots & g_e(-M+3) \\ \vdots & \vdots & \vdots & & \vdots \\ g_e(M-1) & g_e(M-2) & g_e(M-3) & \cdots & g_e(0) \end{pmatrix}$$

$$\begin{pmatrix} g_e(0) & g_e(M-1) & g_e(M-2) & \cdots & g_e(1) \\ g_e(1) & g_e(0) & g_e(M-1) & \cdots & g_e(2) \\ g_e(2) & g_e(1) & g_e(0) & \cdots & g_e(3) \\ \vdots & \vdots & \vdots & & \vdots \\ g_e(M-1) & g_e(M-2) & g_e(M-3) & \cdots & g_e(0) \end{pmatrix}$$

solution by matrix

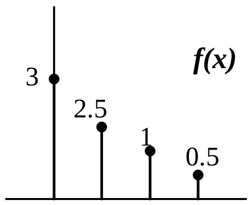
$A=4 \qquad B=4 \qquad then \quad M = A + B - 1 = 7$

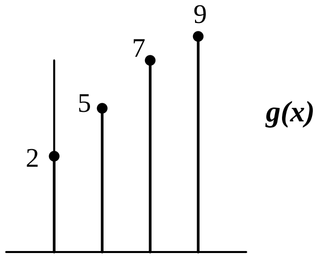
$$\begin{pmatrix} h_e(0) \\ h_e(1) \\ h_e(2) \\ h_e(3) \\ h_e(4) \\ h_e(5) \\ h_e(6) \end{pmatrix} = \begin{pmatrix} g_e(0) & g_e(6) & g_e(5) & g_e(4) & g_e(3) & g_e(2) & g_e(1) \\ g_e(1) & g_e(0) & g_e(6) & g_e(5) & g_e(4) & g_e(3) & g_e(2) \\ g_e(2) & g_e(1) & g_e(0) & g_e(6) & g_e(5) & g_e(4) & g_e(3) \\ g_e(3) & g_e(2) & g_e(1) & g_e(0) & g_e(6) & g_e(5) & g_e(4) \\ g_e(4) & g_e(3) & g_e(2) & g_e(1) & g_e(0) & g_e(6) & g_e(5) \\ g_e(5) & g_e(4) & g_e(3) & g_e(2) & g_e(1) & g_e(0) & g_e(6) \\ g_e(6) & g_e(5) & g_e(4) & g_e(3) & g_e(2) & g_e(1) & g_e(0) \end{pmatrix} \begin{pmatrix} f_e(0) \\ f_e(1) \\ f_e(2) \\ f_e(3) \\ f_e(4) \\ f_e(5) \\ f_e(6) \end{pmatrix}$$

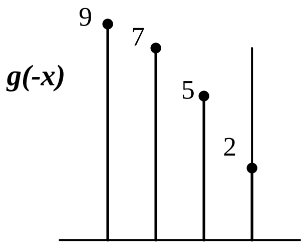
$$= \begin{pmatrix} 2 & 0 & 0 & 0 & 9 & 7 & 5 \\ 5 & 2 & 0 & 0 & 0 & 9 & 7 \\ 7 & 5 & 2 & 0 & 0 & 0 & 9 \\ 9 & 7 & 5 & 2 & 0 & 0 & 0 \\ 0 & 9 & 7 & 5 & 2 & 0 & 0 \\ 0 & 0 & 9 & 7 & 5 & 2 & 0 \\ 0 & 0 & 0 & 9 & 7 & 5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2.5 \\ 1 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

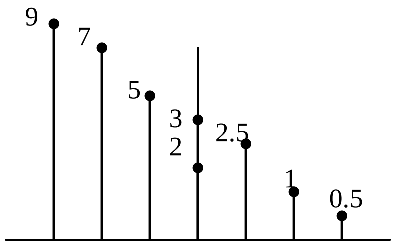
$$= \begin{pmatrix} \mathbf{6} \\ 15 + 5.0 = \mathbf{20} \\ 21 + 12.5 + 2 = \mathbf{35.5} \\ 27 + 17.5 + 5 + 1 = \mathbf{50.5} \\ 22.5 + 7 + 2.5 = \mathbf{32} \\ 9 + 3.5 = \mathbf{12.5} \\ \mathbf{4.5} \end{pmatrix} = \begin{pmatrix} \mathbf{6} \\ \mathbf{20} \\ \mathbf{35.5} \\ \mathbf{50.5} \\ \mathbf{32} \\ \mathbf{12.5} \\ \mathbf{4.5} \end{pmatrix}$$

- ***discrete graphically***

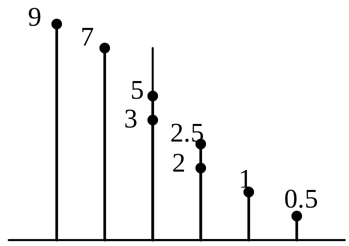




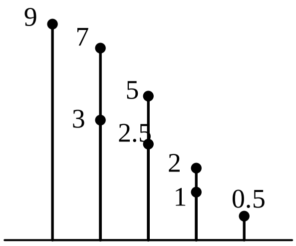




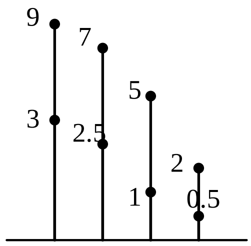
2*3 = ***6***



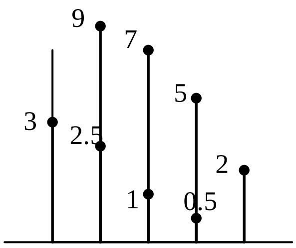
2*2.5+3*5= 5+15= ***20***



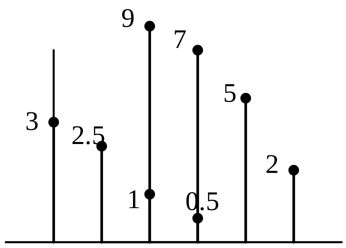
1*2+2.5*5+3*7=2+12.5+21= ***35.5***



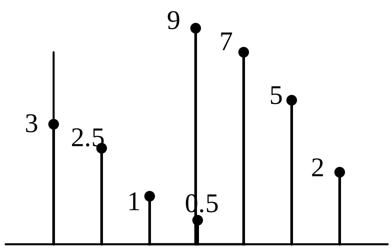
0.5*2+1*5+2.5*7+3*9=1+5+17.5+27=***50.5***



0.5*5+1*7+2.5*9 =2.5+7+22.5= ***32***



1*9+0.5*7 = 9+3.5 = ***12.5***



0.5*9 = ***4.5***