Lecture 6: Numerical Methods for Solving Nonlinear Equations – The Secant Method

The **Secant Method** relies on determining the root of a nonlinear equation by drawing the function's curve and then projecting a series of successive straight lines, determined based on the proposed root values. These straight lines are called **secants** because they intersect the x-axis at specific points.

Concept of the Secant Method

To understand the concept of the Secant Method, assume that we want to find the root of the function $f(x)$. We begin by graphing the function and then selecting two initial guesses for the root, x_1, x_2, with the condition that the function is continuous over the interval $[x_1, x_2]$. There are no other conditions for selecting these two values, meaning $x_1 < x_2$, or vice versa. The algorithm does not require that the interval between the two initial values contains a root of the function, but it is preferable for the interval to contain a root to avoid divergence in the search for the root.

Solution Steps

After selecting x_1, x_2, we define the points $p_1 = (x_1, f(x_1))$ and $p_2 = (x_2, f(x_2))$, then draw a straight line between them $\overrightarrow{p_1 p_2}$. Now, let x_3 be the point where the straight line $\overrightarrow{p_1 p_2}$ intersects the x-axis. If $|x_3 - x_2| \le \varepsilon, |f(x_3)| \le \varepsilon$ or the difference between the proposed root values is within an acceptable error range, we can stop and declare x_3 as the required root.

Otherwise, we update the point $p_3 = \left(x_3, f(x_3)\right)$ and draw a new straight line $\overrightarrow{p_2 p_3}$ and that x_4 is the point of the line of line $\overrightarrow{p_2 p_3}$ with the x-axis,. We continue this process, testing whether x_4 is the root of the function or not from one of the two choices $\left|x_4 - x_3\right| \leq \varepsilon$ or $\left|f\left(x_4\right)\right| \leq \varepsilon$. Thus, we continue to search until one of the two conditions is fulfilled $\left|x_n - x_{n-1}\right| \leq \varepsilon$ or $\left|f\left(x_n\right)\right| \leq \varepsilon$ then x_n is the root required that $n = 3, 4, 5$.......

Deriving the Formula for Root Update

Let x_1, x_2 be the initial guesses for the root of the function $f(x) = 0$. We assume there is a straight line connecting the two points $p_1 = \left(x_1, f(x_1)\right)$ and $p_2 = \left(x_2, f(x_2)\right)$ and intersecting the x-axis at x_3 . From the definition of the slope of a straight line, we can say:

$$slop = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since the slope of the straight line is constant along all points, can be written as:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \quad \text{.................................(1)}$$

Now, since x_3 is a proposed root, we assume that the value of the function at x_3 is zero:

$f(x_3)=0$ Substituting $f(x_3)=0$ into the previous equation and solving for x_3, we obtain the following equation(1) for updating the root:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=\frac{0-f(x_2)}{x_3-x_2}$$

$$\Rightarrow (x_3-x_2)\big(f(x_2)-f(x_1)\big)=(x_2-x_1)\big(-f(x_2)\big)$$

$$\Rightarrow x_3-x_2=-f(x_2)\frac{x_2-x_1}{f(x_2)-f(x_1)}$$

$$\Rightarrow x_3=x_2-f(x_2)\frac{x_2-x_1}{f(x_2)-f(x_1)}$$

General Formula for Updating the Root

In general, the iterative formula for updating the root is:

$$x_n=x_{n-1}-f(x_{n-1})\frac{x_{n-1}-x_{n-2}}{f(x_{n-1})-f(x_{n-2})} \quad \text{……………………..(2)}$$

Where: $n=3,4,5.......$

Secant Method Algorithm:

1. Set the allowed error ε epsilon.
2. Choose two initial guesses for the root x_1, x_2, ensuring the function is continuous over the interval $[x_1, x_2]$.
3. Set $n=3$.
4. Calculate the new root:

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

5. If the condition $|x_n - x_{n-1}| \le \varepsilon$ or $|f(x_n)| \le \varepsilon$ is satisfied, stop and declare x_n as the root.
6. If the condition is not met, set $|x_n - x_{n-1}| > \varepsilon$ and $|f(x_n)| > \varepsilon$, we put $n = n+1$ then return to step 4.

Example: Calculate the root of the equation $f(x) = e^{2x} + 3x$. Use $\varepsilon = 0.0001$.

Solution:

Step 1: The value of ε is specified in the problem $\varepsilon = 0.0001$.

Step 2: Assume $x_1 = -1$, $x_2 = 1$. Notice that the function is continuous over the interval $[-1,1]$ (refer to the graph at the end of the solution).

Step 3:put n=3.

Step 4: Calculate the value of the root according to Equation (2), using n=3.

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}$$

$$= 1 - f(1)\frac{1-(-1)}{f(1) - f(-1)}$$

$$= 1 - 10.3891\frac{2}{10.3891-(-2.8647)}$$

$$= 1 - 10.3891\frac{2}{13.2538}$$

$$= 1\text{-}1.5677$$

$$= -\ 0.5677$$

Notice that:

$$f(-1) = e^{2(-1)} + 3(-1) = 0.1353 - 3 = -2.8647$$

$$f(-1) = e^{2(1)} + 3(1) = 7.3891 + 3 = 10.3891$$

Thus, $x_3 = -0.5677$ is the first proposed value for the root according to the secant algorithm.

Step 5: Test the proposed value from Step 4:

$$f(x_3) = f(-0.5677) = e^{2(-0.5677)} + 3(-0.5677) = 0.3213 - 1.7031 = -1.3818$$

$$|f(x_3)| = |f(-0.5677)| = |-1.3818| = 1.3818 > \varepsilon = 0.0001$$

You can also apply another test:

$$|x_3 - x_2| = |-0.5677 - 1| = 1.5677 > \varepsilon = 0.0001$$

Thus, $x_3 = -0.5677$ is not a root of the above equation.

Step 6: Since the test failed in Step 5, put $n = n+1 = 3+1 = 4$. Then repeat Step 4 as follows:

$$x_4 = x_3 - f(x_3)\frac{x_3 - x_2}{f(x_3) - f(x_2)}$$

$$= -0.5677 - f(-0.5677)\frac{-0.5677 - 1}{f(-0.5677) - f(1)}$$

$$= -0.5677 - \left(-1.3818\right)\frac{-1.5677}{-1.3818 - 10.3891} = -0.3837$$

$$\left|f(x_4)\right| = \left|f(-0.3837)\right| = \left|e^{2(-0.3837)} + 3(-0.3837)\right| = \left|-0.6869\right| > \varepsilon = 0.0001$$

Thus, $x_4 = -0.3837$ is not the desired value. We continue by setting $n = 4 + 1 = 5$

$$x_5 = x_4 - f(x_4)\frac{x_4 - x_3}{f(x_4) - f(x_3)}$$

$$= -0.3837 - f(-0.3837)\frac{-0.3837 - \left(-0.5677\right)}{f(-0.3837) - f(-0.5677)}$$

$$= -0.3837 - \left(-0.6869\right)\frac{0.184}{-0.6869 - \left(-1.3818\right)} = -0.2018$$

$$\left|f(x_5)\right| = \left|f(-0.2018)\right| = \left|e^{2(-0.2018)} + 3(-0.2018)\right| = \left|0.0625\right| > \varepsilon = 0.0001$$

Thus, $x_5 = -0.2018$ is not the desired value. We continue by setting $n = 5 + 1 = 6$:

$$x_6 = x_5 - f(x_5)\frac{x_5 - x_4}{f(x_5) - f(x_4)}$$

$$= -0.2018 - f(-0.2018)\frac{-0.2018 - (-0.3837)}{f(-0.2018) - f(-0.3837)}$$

$$= -0.2018 - 0.0625\frac{0.1819}{0.0625 - (-0.6869)} = -0.217$$

$$|f(x_6)| = |f(-0.217)| = \left|e^{2(-0.217)} + 3(-0.217)\right| = |-0.0031| > \varepsilon = 0.0001$$

Thus, $x_6 = -0.217$ is not the desired value. We continue by setting $n = 6 + 1 = 7$:

$$x_7 = x_6 - f(x_6)\frac{x_6 - x_5}{f(x_6) - f(x_5)}$$

$$= -0.217 - f(-0.217)\frac{-0.217 - (-0.2018)}{f(-0.217) - f(-0.2018)}$$

$$= -0.217 - (-0.0031)\frac{-0.0152}{0.0031 - (-0.0625)} = -0.2163$$

$$|f(x_7)| = |f(-0.2163)| = \left|e^{2(-0.2163)} + 3(-0.2163)\right| = |-0.0001| = \varepsilon$$

Since the absolute value of the function at $x_7 = -0.2163$ has become very small (equal to ε epsilon), we can stop and declare that the estimated root value is -0.2163.

The table below summarizes the results of the steps in the example above

n	x_n	$f(x_n)$	$\lvert f(x_n) \rvert \leq \varepsilon$
1	-1	-2.8647	false
2	1	10.3891	false
3	-0.5677	-1.3818	false
4	-0.3837	-0.6869	false
5	-0.2018	0.0625	false
6	-0.217	-0.0031	false
7	-0.2163	-0.0001	true