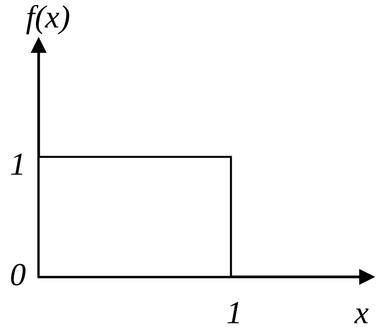
9. *Correlation*

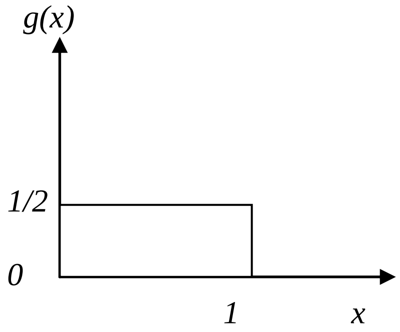
The correlation of two continuous functions *f(x)* and *g(x)* denoted by $f(x)\circ g(x)$ is defined by the relation :

$$f(x)\circ g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x+\alpha)d\alpha$$

This form is similar to convolution the only difference being that the function *g(x)* is not folded about the origin. Thus to perform correlation we simply slide *g(x)* by *f(x)* and integrate the product from $-\infty$ to ∞ for each value of displacement *x*.

<u>Example</u> : find the correlation of the following functions:





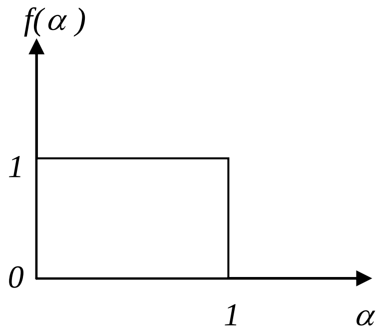
<u>Solution:</u>

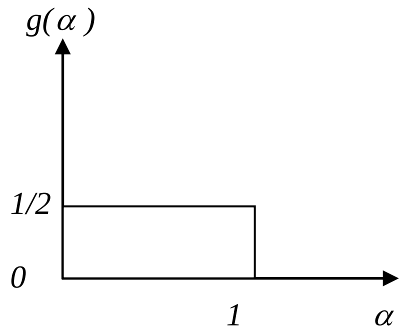
$$f(x)=\begin{cases}1 & 0\le x\le 1\\ 0 & otherwise\end{cases} \qquad g(x)=\begin{cases}1/2 & 0\le x\le 1\\ 0 & otherwise\end{cases}$$

$$f(x)\circ g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x+\alpha)d\alpha$$

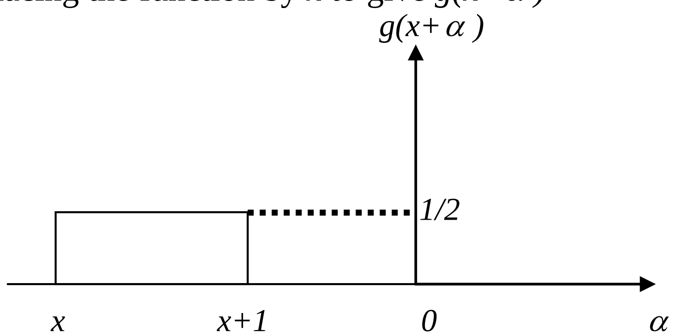
- ***graphically***

step1: change the axis





Step2: displacing the function by *x* to give *g(x+α)*



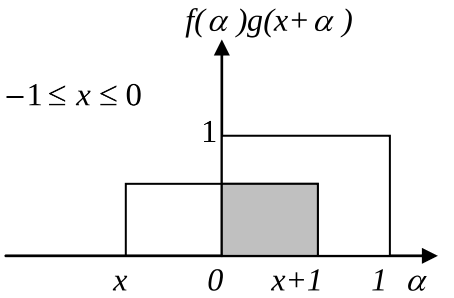
Step3: Sliding , for any given value of x , we multiply $f(\alpha)$ by the corresponding $g(x+\alpha)$ and integrate the product from $-\infty$ *to* ∞.

when: $0 \le x+1 \le 1$

$-1 \le x \le 0$

$$\int_0^{x+1} f(\alpha)g(x+\alpha)d\alpha$$

$$= \int_0^{x+1} 1.\frac{1}{2}d\alpha \quad = \frac{1}{2}\alpha\Big|_0^{x+1} \quad = \frac{x+1}{2}$$

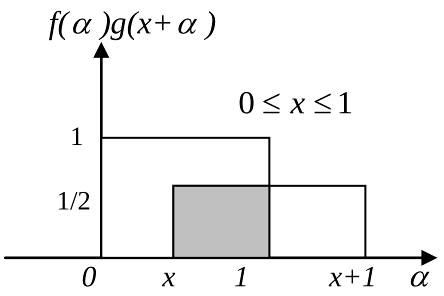


when :

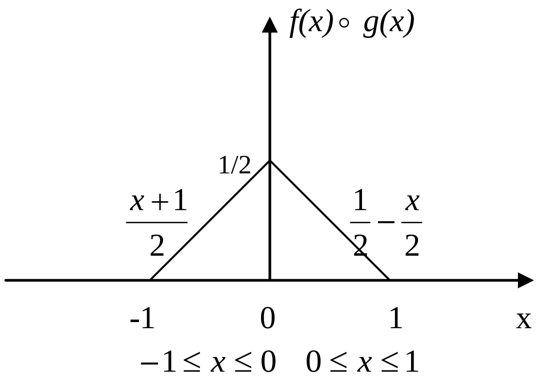
$1 \le x+1 \le 2$

$0 \le x \le 1$

$$\int_x^1 \frac{1}{2}.1d\alpha \quad = \frac{1}{2}\alpha\Big]_x^1 \quad = \frac{1}{2} - \frac{x}{2}$$



$$f(x) \circ g(x) = \begin{cases} \dfrac{x+1}{2} & -1 \le x \le 0 \\ \dfrac{1}{2} - \dfrac{x}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



- ***mathematically***

$$f(x)\circ g(x)=\int_{-\infty}^{\infty} f(\alpha)g(x+\alpha)d\alpha \qquad 0\le\alpha\le 1 \quad \text{for} \quad f(\alpha)$$

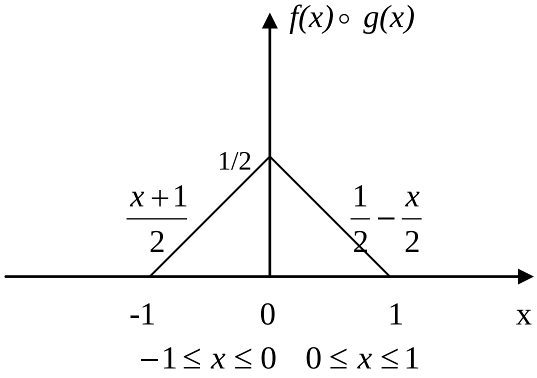
$$0\le\alpha\le 1 \quad \text{for} \quad g(\alpha)$$

$$0\le\alpha\le 1$$
$$x\le\alpha\le x+1$$

$$\int_{0}^{x+1} f(\alpha)g(x+\alpha)d\alpha \;=\; \int_{0}^{x+1} 1.\frac{1}{2}d\alpha \;=\; \frac{1}{2}\alpha\Big|_{0}^{x+1} \;=\; \frac{x+1}{2}$$

$$\int_{x}^{1}\frac{1}{2}.1d\alpha \;=\; \frac{1}{2}\alpha\Big]_{x}^{1} \;=\; \frac{1}{2}-\frac{x}{2}$$

$$f(x)\circ g(x)=\begin{cases}\dfrac{x+1}{2} & -1\le x\le 0\\[2mm] \dfrac{1}{2}-\dfrac{x}{2} & 0\le x\le 1\\[2mm] 0 & \text{otherwise}\end{cases}$$



The discrete equivalent of convolution eq. is defined as :

$$f_e(x)\circ g_e(x)=\sum_{m=0}^{M-1} f_e(m)g_e(x+m)$$

for $x=0,1,2,\ \ldots\ ,\ M-1$. The correlation function is a discrete, periodic array of length *M* , with the values *$x=0,1,2\ ,\ \ldots\ ,\ M-1$* describing a full period of $f_e(x)\circ g_e(x)$.

<u>Example</u>: Find the correlation function of the following functions:

f = { 3 , 2.5 , 1 , 0.5 }
g = { 2 , 5 , 7 , 9 }

<u>solution:</u>
f contains 4 elements
g contains 4 elements
M = A+ B – 1 = 4 + 4 – 1 = 7 the interval 0 . . . 6

$$h_e(x) = f_e(x) \circ g_e(x) = \sum_{m=0}^{M-1} f_e(m) g_e(x+m)$$

$$\begin{aligned}
h_e(0) &= f_e(0).g_e(0) + f_e(1).g_e(1) + f_e(2).g_e(2) + f_e(3).g_e(3) \\
&\quad + f_e(4).g_e(4) + f_e(5).g_e(5) + f_e(6).g_e(6) \\
&= 3 * 2 + 2.5 * 5 + 1 * 7 + 0.5 * 9 = 6 + 12.5 + 7 + 4.5 \mathbf{= 30}
\end{aligned}$$

$$\begin{aligned}
h_e(1) &= f_e(0).g_e(1) + f_e(1).g_e(2) + f_e(2).g_e(3) + f_e(3).g_e(4) \\
&\quad + f_e(4).g_e(5) + f_e(5).g_e(6) + f_e(6).g_e(0) \\
&= 3 * 5 + 2.5 * 7 + 1 * 9 + 0.5 * 0 = 15 + 17.5 + 9 = \mathbf{41.5}
\end{aligned}$$

$$\begin{aligned}
h_e(2) &= f_e(0).g_e(2) + f_e(1).g_e(3) + f_e(2).g_e(4) + f_e(3).g_e(5) \\
&\quad + f_e(4).g_e(6) + f_e(5).g_e(0) + f_e(6).g_e(1) \\
&= 3 * 7 + 2.5 * 9 + 1 * 0 = 21 + 22.5 = 4\mathbf{3.5}
\end{aligned}$$

$$\begin{aligned}
h_e(3) &= f_e(0).g_e(3) + f_e(1).g_e(4) + f_e(2).g_e(5) + f_e(3).g_e(6) \\
&\quad + f_e(4).g_e(0) + f_e(5).g_e(1) + f_e(6).g_e(2) \\
&= 3 * 9 + 2.5 * 0 + 1 * 0 + 0.5 * 0 + 0*2 + 0*5 + 0*7 = \mathbf{27}
\end{aligned}$$

$$\begin{aligned}
h_e(4) &= f_e(0).g_e(4) + f_e(1).g_e(5) + f_e(2).g_e(6) + f_e(3).g_e(0) \\
&\quad + f_e(4).g_e(1) + f_e(5).g_e(2) + f_e(6).g_e(3) \\
&= 3 * 0 + 2.5 * 0 + 1 * 0 + 0.5 * 2 + 0 * 5 + 0 * 7 + 0*9 = \mathbf{1.0}
\end{aligned}$$

$$\begin{aligned}
h_e(5) &= f_e(0).g_e(5) + f_e(1).g_e(6) + f_e(2).g_e(0) + f_e(3).g_e(1) \\
&\quad + f_e(4).g_e(2) + f_e(5).g_e(3) + f_e(6).g_e(4) \\
&= 3 * 0 + 2.5 * 0 + 1 * 2 + 0.5 * 5 + 0 * 7 + 0*9 = 2 + 2.5 = \mathbf{4.5}
\end{aligned}$$

$$\begin{aligned}
h_e(6) &= f_e(0).g_e(6) + f_e(1).g_e(0) + f_e(2).g_e(1) + f_e(3).g_e(2) \\
&\quad + f_e(4).g_e(3) + f_e(5).g_e(4) + f_e(6).g_e(5) \\
&= 3*0 + 2.5 * 2 + 1 * 5 + 0.5 * 7 + 0 * 9 = 5.0 + 5 + 3.5 \mathbf{= 13.5}
\end{aligned}$$

-discrete correlation by matrix

$$\begin{pmatrix}
g_e(0) & g_e(1) & g_e(2) & g_e(3) & \cdots & g_e(M-1) \\
g_e(1) & g_e(2) & g_e(3) & g_e(4) & \cdots & g_e(0) \\
g_e(2) & g_e(3) & g_e(4) & g_e(5) & \cdots & g_e(1) \\
g_e(3) & g_e(4) & g_e(5) & g_e(6) & \cdots & g_e(2) \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
g_e(M-1) & g_e(0) & g_e(1) & g_e(2) & \cdots & g_e(M-2)
\end{pmatrix}$$

- ***solution by matrix:***

$A=4 \qquad B=4 \qquad then \quad M = A + B - 1 = 7$

$$\begin{pmatrix} h_e(0) \\ h_e(1) \\ h_e(2) \\ h_e(3) \\ h_e(4) \\ h_e(5) \\ h_e(6) \end{pmatrix} = \begin{pmatrix} g_e(0) & g_e(1) & g_e(2) & g_e(3) & g_e(4) & g_e(5) & g_e(6) \\ g_e(1) & g_e(2) & g_e(3) & g_e(4) & g_e(5) & g_e(6) & g_e(0) \\ g_e(2) & g_e(3) & g_e(4) & g_e(5) & g_e(6) & g_e(0) & g_e(1) \\ g_e(3) & g_e(4) & g_e(5) & g_e(6) & g_e(0) & g_e(1) & g_e(2) \\ g_e(4) & g_e(5) & g_e(6) & g_e(0) & g_e(1) & g_e(2) & g_e(3) \\ g_e(5) & g_e(6) & g_e(0) & g_e(1) & g_e(2) & g_e(3) & g_e(4) \\ g_e(6) & g_e(0) & g_e(1) & g_e(2) & g_e(3) & g_e(4) & g_e(5) \end{pmatrix} \begin{pmatrix} f_e(0) \\ f_e(1) \\ f_e(2) \\ f_e(3) \\ f_e(4) \\ f_e(5) \\ f_e(6) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 & 7 & 9 & 0 & 0 & 0 \\ 5 & 7 & 9 & 0 & 0 & 0 & 2 \\ 7 & 9 & 0 & 0 & 0 & 2 & 5 \\ 9 & 0 & 0 & 0 & 2 & 5 & 7 \\ 0 & 0 & 0 & 2 & 5 & 7 & 9 \\ 0 & 0 & 2 & 5 & 7 & 9 & 0 \\ 0 & 2 & 5 & 7 & 9 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2.5 \\ 1 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{30} \\ \mathbf{41.5} \\ \mathbf{43.5} \\ \mathbf{27} \\ \mathbf{1} \\ \mathbf{4.5} \\ \mathbf{13.5} \end{pmatrix}$$