After satisfying the stationarity, the next step includes characterizing time series model's order. Studying models characteristics is important to characterize time series model's orders. Autoregressive (AR), moving average (MA), and autoregressive integrated moving average (ARIMA) models can be presented briefly as follows.

(a) Autoregressive Model

An autoregressive process can be employed to express the current time series value using linear regression function of *p* previous time series values. *p*-th order autoregressive AR(*p*) can be written in general as follows.

$$\phi(B)Z_t = a_t \quad (1)$$

$$(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)Z_t = a_t \quad (2)$$

or

$$Z_t = \phi_1 Z_{t-1}+\phi_2 Z_{t-2}+\cdots+\phi_p Z_{t-p}+a_t \quad (3)$$

where ϕ_k is the k^{th} autoregressive coefficient or parameter that reflect the effect of changing the k^{th} autoregressive variable Z_{t-k} on Z_t in the autoregressive model, $k=1,2,3,\ldots,p$, a_t is the independent and identically distributed random error that represents white noise process with mean zero and constant variable σ_a^2 or can be written as $a_t \sim i.i.d.N(0,\sigma_a^2)$, and $\phi(B)=(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)$.

If the time series data needs any order of differencing to satisfy the mean stationary, $W_t=(1-B)^d Z_t$, W_t is written instead of Z_t in the equations (1), (2), and (3). The most important characteristics of AR(*p*) are listed below.

i. The variance of AR(*p*) is written as follows.

$$\gamma_0=\sigma_Z^2=\sigma_a^2 \big/ (1-\phi_1\rho_1-\phi_2\rho_2-\cdots-\phi_p\rho_p). \quad (4)$$

ii. The autocorrelation of AR(*p*) is as follows.

$$\rho_k=\phi_1\rho_{k-1}-\phi_2\rho_{k-2}-\cdots-\phi_p\rho_{k-p} \quad (5)$$

where $k=1,2,3,\ldots$.

iii. The partial autocorrelation of AR(*p*) can be just evaluated as significant or insignificant such as follows.

$$\phi_{kk}\neq 0\ ; k\le p \quad (6)$$

$$\phi_{kk} = 0 \ ; k > p \ . \tag{7}$$

(b) Moving Average Model

A moving average process can be employed to express the current time series value Z_t using linear regression function of q previous random errors. q-th order moving average MA(q) can be written in general as follows.

$$Z_t = \theta(B)a_t \tag{8}$$

$$Z_t = (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t \tag{9}$$

or

$$Z_t = -\theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} + a_t \tag{10}$$

where θ_k is the k^{th} moving average parameter that reflects the effect of changing the k^{th} random error a_{t-k} on Z_t in the moving average model, $k = 1, 2, 3, \ldots, q$, a_t is the independent and identically distributed random error that represents white noise process with mean zero and constant variable σ_a^2 or can be written as $a_t \sim i.i.d.N(0, \sigma_a^2)$, and $\theta(B) = (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_p B^p)$.

If the time series data needs any order of differencing to satisfy the mean stationarity, $W_t = (1-B)^d Z_t$, W_t is written instead of Z_t in the equations (8), (9), (10). The most important characteristics of MA(q) are listed below.

i. The variance of MA(q) is written as follows.

$$\gamma_0 = \sigma_Z^2 = (1+\theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_a^2 \ . \tag{11}$$

ii. The autocorrelation of MA(q) is as follows.

$$\rho_k = \frac{-\theta_k + \sum_{i=1}^{q-k} \theta_i \theta_{k+i}}{(1+\theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)} \tag{12}$$

where $k = 1, 2, 3, \ldots, q$.

and

$$\rho_k = 0 \tag{13}$$

where $k > q$.

iii. The partial autocorrelation function of MA(q) decays toward zero.

(c) ARIMA Models

The nonstationary mean of time series data requires taking sequence differences one or more times for the original nonstationary series. Time series model may include AR(p) components, MA(q) components, or all of them together. When AR(p) and MA(q) components are combined in one model, this model can be called ARMA(p,q) for stationary data or ARIMA(p,d,q) after performing d of differencing processes. In general, ARIMA(p,d,q) model is written such as the following equation.

$$\phi(B)(1-B)^d Z_t = \theta(B)a_t \tag{14}$$

$$\phi(B)W_t = \theta(B)a_t \tag{15}$$

where $W_t = (1-B)^d Z_t$.

$$(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)W_t = (1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)a_t \tag{16}$$

$$W_t = \phi_1 W_{t-1}+\phi_2 W_{t-2}+\cdots+\phi_p W_{t-p}-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\cdots-\theta_q a_{t-q}+a_t \tag{17}$$

where ϕ_k and θ_k are the k^{th} autoregressive and moving average parameters, respectively, that reflect the effects of changing the k^{th} time series variable Z_{t-k} and random error a_{t-k}, respectively, on Z_t, while a_t is the independent and identically distributed random error that represents white noise process with mean zero and constant variable σ_a^2 or can be written as $a_t \sim i.i.d.N(0,\sigma_a^2)$, $\phi(B)=(1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p)$, and $\theta(B)=(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_p B^p)$.

The characteristics of ARIMA(p,d,q) include the combination of the stationarity and the invertibility characteristics for AR(p) and MA(q). AR(p) and MA(q) are regarded as special cases of ARIMA(p,d,q), AR(p) can be symbolized as ARIMA(p,0,0) and MA(q) can be written as ARIMA(0,0,q). Seasonal time series data can be generalized according to the rules mentioned in the previous sections. For example, a multiplicative seasonal ARIMA model or ARIMA(p,d,q)(P,D,Q)$_s$ can be written in general such as follows.

$$\phi(B)\Phi(B)(1-B)^D(1-B)^d Z_t = \theta(B)\Theta(B)a_t \tag{18}$$

$$\phi(B)\Phi(B)W_t = \theta(B)\Theta(B)a_t \tag{19}$$

where

$$W_t = (1-B)^D (1-B)^d Z_t \ ,$$

$$\phi(B) = (1-\phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) \ ,$$

$$\Phi(B) = (1-\Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps}) \ ,$$

$$\theta(B) = (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) \ ,$$

and

$$\Theta(B) = (1-\Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{Qs}) \ .$$

After plotting time series data and satisfying the stationarity of mean and variance as mentioned previously, specifying the orders p and q is the most important procedure for model identification. From the characteristics of time series models mentioned previously, using ACF and PACF is useful to specify the suitable time series model and the orders p and q such as in Table 3.1.

Table 3.1 : ACF and PACF patterns according to time series models

Model	**ACF**	**PACF**
AR(p)	Dies out	Cuts off after lag p
MA(q)	Cuts off after lag q	Dies out
ARIMA(p,d,q)	Dies out but goes to zero after lag q	Dies out but goes to zero after lag p