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Probability distributions: A probability distribution is a mathematical function that describes the probability of different possible values of a variable. Probability distributions are often depicted using graphs or probability tables.

Discrete probability distributions: A random variable X is said to have a discrete distribution if the sample space of X is countable.

Discrete uniform distribution: A random variable X is said to have a discrete *uniform*(k) distribution, denoted by $X \sim Uniform(N)$, if its PMF has the following form:

$$f(x) = \begin{cases} \frac{1}{N} & x = 1, 2, ..., N \\ 0 & \text{otherwise} \end{cases}$$

where N is a specified positive integer, and it is called the parameter of the distribution. This distribution puts equal mass on each of the possible outcomes $1, 2, ..., N$.

Mean and variance: By definition of the mean and variance, we have

$$E[X] = \sum_{x=1}^{N} x f(x) = \frac{1}{N} \sum_{x=1}^{N} x = \frac{1}{N} \left(\frac{N(N+1)}{2} \right) = \frac{N+1}{2}$$

$$E[X^2] = \sum_{x=1}^{N} x^2 f(x) = \frac{1}{N} \sum_{x=1}^{N} x^2 = \frac{1}{N} \left(\frac{N(N+1)(2N+1)}{6} \right) = \frac{(N+1)(2N+1)}{6}$$

$$
\begin{aligned}
\operatorname{Var}(X) &= E[X^2] - E[X] \\
&= \frac{(N+1)(2N+1)}{6} - \left(\frac{N+1}{2}\right)^2 \\
&= \frac{N+1}{2}\left(\frac{2N+1}{3} - \frac{N+1}{2}\right) \\
&= \frac{N+1}{2}\left(\frac{2(2N+1) - 3(N+1)}{6}\right) \\
&= \frac{N+1}{2}\left(\frac{4N+2-3N-3}{6}\right) \\
&= \frac{(N+1)(N-1)}{12} = \frac{N^2-1}{12}
\end{aligned}
$$

Moment generating function:

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \frac{1}{N}\sum_{x=1}^{N} e^{tx} \\
&= \frac{1}{N}\left[e^t + e^{2t} + e^{3t} + \cdots + e^{Nt}\right] \\
&= \frac{1}{N}\left[\left(e^t\right)^1 + \left(e^t\right)^2 + \left(e^t\right)^3 + \cdots + \left(e^t\right)^N\right] \\
&= \frac{1}{N}e^t\left[\left(e^t\right)^0 + \left(e^t\right)^1 + \left(e^t\right)^2 + \cdots + \left(e^t\right)^{N-1}\right] \\
&= \frac{1}{N}e^t\sum_{x=0}^{N-1}\left(e^t\right)^x \\
&= \frac{1}{N}\left(\frac{e^t\left(1-e^{Nt}\right)}{1-e^t}\right)
\end{aligned}
$$

notice that for any $r \neq 1$ the following geometric series can be written as:

$$\begin{aligned} s_N &= ar^0 + ar^1 + ar^2 + \cdots + ar^{N-1} \\ &= a\left(\frac{1-r^N}{1-r}\right) \end{aligned}$$

Cumulative distribution function: The CDF of a discrete uniform random variable is given by

$$F(x) = \sum_{k=1}^{x} f(k) = \sum_{k=1}^{x} \frac{1}{N} = \frac{1}{N}\sum_{k=1}^{x}(1) = \frac{1}{N}(1 + 1 + \cdots + 1) = \frac{x}{N}$$

hence

$$F(x) = \begin{cases} \frac{x}{N} & x = 1, 2, ..., N \\ 1 & x \geq N \\ 0 & \text{otherwise} \end{cases}$$

Example 1.1 (Homework): Let $X \sim uniform(N)$. Use the MGF of X to find the mean and variance of X.

Example 1.2 (Homework): Let $X \sim uniform(5)$

1- Write down the PMF of X.

2- Find the mean, the variance, the CDF, and the MGF of X.

3- Find $P(2 < X \leq 4)$.

Example 1.3: If $X \sim Uniform(N)$, find the value of N, such that $P(X \leq E[X]) = 0.65$.

Solution: We have $E[X] = \frac{N+1}{2}$. Hence

$$P(X \leq E[X]) = P\left(X \leq \frac{N+1}{2}\right)$$
$$= F\left(\frac{N+1}{2}\right)$$

We also have $F(x) = \frac{x}{N}$, so that

$$F\left(\frac{N+1}{2}\right) = \frac{(N+1)/2}{N}$$
$$= \frac{N+1}{2N}$$

therefore

$$P(X \leq E[X]) = 0.65$$
$$\implies \frac{N+1}{2N} = 0.65$$
$$\implies N+1 = 1.3N$$
$$\implies 1.3N - N = 1$$
$$\implies 0.3N = 1$$
$$\implies N = \frac{1}{0.3} \implies N \approx 3$$

Find N such that $F(x) \leq \frac{N}{2}$ **(Homework)**.

Bernoulli distribution: A random variable X is said to have a $Bernoulli(p)$ distribution, denoted by $X \sim Bernoulli(p)$, if its PMF has the following form:

$$f(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \text{ and } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where p is called the probability of success ($1-p$ here is called the probability of failure). This distribution is used with the experiments that have only two possible outcomes. If $X = 1$, then the experiment turned a "success". While if $X = 0$, then the experiment turned a "failure".

Mean and variance: By definition of the mean and variance, we have

$$E[X] = \sum_{x=0}^{1} xf(x) = \sum_{x=0}^{1} xp^x(1-p)^{1-x} = (0)p^0(1-p)^{1-0} + (1)p^1(1-p)^{1-1} = p$$

$$E[X^2] = \sum_{x=0}^{1} x^2f(x) = \sum_{x=0}^{1} x^2p^x(1-p)^{1-x} = (0)p^0(1-p)^{1-0} + (1)p^1(1-p)^{1-1} = p$$

notice that the rth moment about the origin is $E[X^r] = p$, for $r = 1, 2, 3, ...$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X] \\ &= p - p^2 = p(1-p) \end{aligned}$$

Moment generating function:

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} \\ &= \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} \\ &= e^{t(0)} p^0 (1-p)^{1-0} + e^{t(1)} p^1 (1-p)^{1-1} \\ &= e^t p + (1-p) \end{aligned}$$

Find the characteristic function of $X \sim Bernoulli(p)$ **(Homework)**

Example 1.4: Let X_1 and X_2 have the following joint probability function:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} p^{x_1+x_2}(1-p)^{2-x_1-x_2} & x_1, x_2 = 0, 1 \text{ and } 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1- Show that $f_{X_1,X_2}(x_1,x_2)$ is a joint PMF.

2- Find the marginal PMFs.

3- Find the conditional PMFs, and check if X_1 and X_2 are dependent or not.

Solution: 1- Show that $f_{X_1,X_2}(x_1,x_2)$ is a joint PMF

$$\sum_{x_2=0}^{1}\sum_{x_1=0}^{1} f_{X_1,X_2}(x_1,x_2) = \sum_{x_2=0}^{1}\sum_{x_1=0}^{1} p^{x_1+x_2}(1-p)^{2-x_1-x_2}$$
$$= (1-p)^2 + p(1-p) + p(1-p) + p^2$$
$$= (1-p)\left((1-p)+p+p\right) + p^2$$
$$= (1-p)(1+p) + p^2$$
$$= 1 - p^2 + p^2 = 1$$

therefore $f_{X_1,X_2}(x_1,x_2)$ is a joint PMF.

2- The marginals

$$f_{X_1}(x_1) = \sum_{x_2=0}^{1} f_{X_1,X_2}(x_1,x_2)$$
$$= \sum_{x_2=0}^{1} p^{x_1+x_2}(1-p)^{2-x_1-x_2}$$
$$= p^{x_1+0}(1-p)^{2-x_1-0} + p^{x_1+1}(1-p)^{2-x_1-1}$$
$$= p^{x_1}(1-p)^{2-x_1} + p^{x_1+1}(1-p)^{1-x_1}$$
$$= p^{x_1}(1-p)^{1-x_1}\left((1-p)+p\right) = p^{x_1}(1-p)^{1-x_1}$$

so that $X_1 \sim Bernoulli(p)$

$$f_{X_1}(x_1) = \begin{cases} p^{x_1}(1-p)^{1-x_1} & x_1 = 0,1 \text{ and } 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

show that $X_2 \sim Bernoulli(p)$ **(Homework)**.

3- The conditional PMFs:

$$\begin{aligned} f_{X_2|X_1}(x_2|x_1) &= \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} \\ &= \frac{p^{x_1+x_2}(1-p)^{2-x_1-x_2}}{p^{x_1}(1-p)^{1-x_1}} \\ &= p^{x_1+x_2}(1-p)^{2-x_1-x_2}p^{-x_1}(1-p)^{-1+x_1} \\ &= p^{x_1+x_2-x_1}(1-p)^{2-x_1-x_2-1+x_1} = p^{x_2}(1-p)^{1-x_2} \end{aligned}$$

hence

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} p^{x_2}(1-p)^{1-x_2} & x_2 = 0,1 \text{ and } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

similarly **(Homework)**

$$f_{X_1|X_2}(x_1|x_2) = \begin{cases} p^{x_1}(1-p)^{1-x_1} & x_1 = 0,1 \text{ and } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

since

$$f_{X_2|X_1}(x_2|x_1) = f_{X_2}(x_2) = p^{x_2}(1-p)^{1-x_2}$$

$$f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1) = p^{x_1}(1-p)^{1-x_1}$$

we can say that X_1 and X_2 are independent. Notice that we can prove that X_1 and X_2 are independent by

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

Example 1.5: Let $X \sim Bernoulli(0.97)$.

1- Find the mean, the variance, and the MGF of $Y = 2X + 3$.

2- Then find $P(Y \leq 4)$

Solution:

$$\begin{aligned} E[Y] &= E[2X + 3] \\ &= 2E[X] + 3 = 2(0.97) + 3 = 4.94 \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(2X + 3) \\ &= 4\text{Var}(X) \\ &= 4(0.97)(1 - 0.97) = 0.1164 \end{aligned}$$

$$
\begin{aligned}
M_{(Y)}(t) &= E[e^{tY}] \\
&= E[e^{t(2X+3)}] \\
&= E[e^{2tX+3t}] \\
&= e^{3t}E[e^{2tX}] \\
&= e^{3t}M_{(X)}(2t) \\
&= e^{3t}\left(0.97e^{2t} + (1-0.97)\right) \\
&= 0.97e^{5t} + 0.03e^{3t}
\end{aligned}
$$

find $P(Y \leq 4)$

$$
\begin{aligned}
P(Y \leq 4) &= P(2X+3 \leq 4) \\
&= P(X \leq \frac{4-3}{2}) \\
&= P(X \leq 1/2) \\
&= P(X = 0) \\
&= 0.97^0(1-0.97)^{1-0} = 0.03
\end{aligned}
$$

Binomial distribution: A random variable Y is said to have a *Binomial*(n, p) distribution, denoted by $Y \sim Binomial(n, p)$, if its PMF has the following form:

$$f(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y} & y = 0, 1, ..., n \text{ and } 0 \leq p \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where p is called the probability of success ($1 - p$ here is called the probability of failure), and n is a positive integer represents the total number of trials. **Here Y represents the number of successes out of n independent Bernoulli trials**. That is, if $X_1, X_2, ..., X_n$ are n independent Bernoulli random variables (or Bernoulli trials), each takes the values 0 or 1, then $Y = \sum_{j=1}^{n} X_j$ has a binomial distribution with parameters n and p.

The binomial expansion: For any two real numbers a and b, the polynomial $(a + b)^n$ can be written as:

$$(a+b)^n = \sum_{y=0}^{n} \binom{n}{y} a^y b^{n-y}$$

now if $a = p$ and $b = 1 - p$, then

$$(p + (1-p))^n = \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y}$$

$$\implies 1 = \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y}$$

Mean and variance: By definition of the mean and variance, we have

$$
\begin{aligned}
E[Y] &= \sum_{y=0}^{n} y f(y) \\
&= \sum_{y=0}^{n} y \binom{n}{y} p^y (1-p)^{n-y} \\
&= \sum_{y=1}^{n} y \binom{n}{y} p^y (1-p)^{n-y}, \quad \text{because the first term is zero} \\
&= \sum_{y=1}^{n} y \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} \\
&= \sum_{y=1}^{n} y \frac{n(n-1)!}{y(y-1)!(n-y)!} p^y (1-p)^{n-y} \\
&= np \sum_{y=1}^{n} \frac{(n-1)!}{(y-1)!(n-y)!} p^{y-1} (1-p)^{n-y} \\
&= np \sum_{y=1}^{n} \binom{n-1}{y-1} p^{y-1} (1-p)^{n-y}
\end{aligned}
$$

let $k = y - 1 \implies y = k + 1$, then k takes the values $0, 1, 2, ..., n-1$.

Then

$$
\begin{aligned}
E[Y] &= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&= np \left(p + (1-p)\right)^{n-1}, \quad \text{by the binomial expansion} \\
&= np
\end{aligned}
$$

Now to find $E[Y^2]$, we calculate $E[Y(Y-1)] = E[Y^2] - E[Y]$, then $E[Y^2] = E[Y(Y-1)] + E[Y]$

$$
\begin{aligned}
E[Y(Y-1)] &= \sum_{y=0}^{n} y(y-1)f(y) \\
&= \sum_{y=0}^{n} y(y-1)\binom{n}{y}p^y(1-p)^{n-y} \\
&= \sum_{y=2}^{n} y(y-1)\binom{n}{y}p^y(1-p)^{n-y}, \quad \text{because the first two terms are zeros} \\
&= \sum_{y=2}^{n} y(y-1)\frac{n!}{y!(n-y)!}p^y(1-p)^{n-y} \\
&= \sum_{y=2}^{n} y(y-1)\frac{n(n-1)(n-2)!}{y(y-1)(y-2)!(n-y)!}p^y(1-p)^{n-y} \\
&= n(n-1)p^2\sum_{y=2}^{n} \frac{(n-2)!}{(y-2)!(n-y)!}p^{y-2}(1-p)^{n-y} \\
&= n(n-1)p^2\sum_{y=2}^{n} \binom{n-2}{y-2}p^{y-2}(1-p)^{n-y}
\end{aligned}
$$

let $k = y-2 \implies y = k+2$, then k takes the values $0, 1, 2, ..., n-2$.

Then

$$
\begin{aligned}
E[Y(Y-1)] &= n(n-1)p^2\sum_{k=0}^{n-2} \binom{n-2}{k}p^k(1-p)^{n-2-k} \\
&= n(n-1)p^2\left(p+(1-p)\right)^{n-2}, \quad \text{by the binomial expansion} \\
&= n(n-1)p^2 = n^2p^2 - np^2
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] &= E[Y(Y-1)] + E[Y] \\
&= n^2p^2 - np^2 + np
\end{aligned}
$$

$$\begin{aligned}\text{Var}(Y) &= E[Y^2] - E[Y] \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= np(1-p)\end{aligned}$$

Moment generating function:

$$\begin{aligned}M_Y(t) &= E[e^{tY}] \\ &= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y (1-p)^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} \left(pe^t\right)^y (1-p)^{n-y} \\ &= \left(pe^t + (1-p)\right)^n, \quad \text{by the binomial expansion}\end{aligned}$$

Example 1.6: Let Y be a discrete random variable with the following MGF:

$$M_Y(t) = (0.6e^t + 0.4)^3$$

1- What is the distribution of Y? Write down the PMF of Y.

2- Use the MGF to find the mean and variance of Y. Then find the mean and variance of $W = \frac{Y - E[Y]}{\sqrt{\text{Var}(Y)}}$.

3- Find the MGF of W.

Solution:

1- The distribution of Y is binomial with $n = 3$ and $p = 0.6$; That is, Y has the following PMF

$$f(y) = \begin{cases} \binom{3}{y} 0.6^y 0.4^{3-y} & y = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X] &= \frac{d}{dt} M_X(t) \bigg|_{t=0} \\ &= \frac{d}{dt} (0.6e^t + 0.4)^3 \bigg|_{t=0} \\ &= 3(0.6e^t + 0.4)^2 (0.6e^t) \big|_{t=0} \\ &= 1.8e^0 (0.6e^0 + 0.4)^2 \\ &= 1.8 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \frac{d^2}{dt^2} M_X(t) \bigg|_{t=0} \\ &= \frac{d}{dt} 1.8e^t (0.6e^t + 0.4)^2 \bigg|_{t=0} \\ &= 1.8e^t (2(0.6e^t + 0.4)(0.6e^t)) + 1.8e^t (0.6e^t + 0.4)^2 \big|_{t=0} \\ &= 1.8e^0 (2(0.6e^0 + 0.4)(0.6e^0)) + 1.8e^0 (0.6e^0 + 0.4)^2 \\ &= 1.8(2(0.6)) + 1.8 = 3.96 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ &= 3.96 - (1.8)^2 = 0.72 \end{aligned}$$

Now the mean and variance of $W = \frac{Y - E[Y]}{\sqrt{\text{Var}(Y)}}$

$$E[W] = E\left[\frac{Y - E[Y]}{\sqrt{\text{Var}(Y)}}\right]$$
$$= E\left[\frac{Y - 1.8}{0.72}\right]$$
$$= \frac{E[Y] - 1.8}{0.72}$$
$$= \frac{1.8 - 1.8}{0.72} = 0$$

$$\text{Var}(W) = \text{Var}\left(\frac{Y - E[Y]}{\sqrt{\text{Var}(Y)}}\right)$$
$$= \text{Var}\left(\frac{Y - 1.8}{\sqrt{0.72}}\right)$$
$$= \frac{1}{0.72}\left(\text{Var}\left(Y\right) + \text{Var}\left(1.8\right)\right)$$
$$= \frac{1}{0.72}\left(0.72 + 0\right) = 1$$

MGF of W

$$
\begin{aligned}
M_W(t) &= E[e^{tW}] \\
&= E\left[e^{t\left(\frac{Y-E[Y]}{\sqrt{\text{Var}(Y)}}\right)}\right] \\
&= E\left[e^{t\left(\frac{Y-1.8}{\sqrt{0.72}}\right)}\right] \\
&= E\left[e^{-\frac{1.8t}{\sqrt{0.72}}} e^{\frac{t}{\sqrt{0.72}}Y}\right] \\
&= e^{-\frac{1.8t}{\sqrt{0.72}}} E\left[e^{\frac{t}{\sqrt{0.72}}Y}\right] \\
&= e^{-\frac{1.8t}{\sqrt{0.72}}} M_Y\left(t/\sqrt{0.72}\right) \\
&= e^{-\frac{1.8t}{\sqrt{0.72}}} \left(0.6e^{t/\sqrt{0.72}} + 0.4\right)^3
\end{aligned}
$$

Example 1.7: Write down the prbability functions and MGFs of each of the following distributions:

1- $X \sim Uniform(5)$

$$
f(x) = \begin{cases} \frac{1}{5} & x = 1, 2, ..., 5 \\ \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
M_X(t) &= \frac{1}{5}(e^t + e^{2t} + \cdots + e^{5t}) \\
&= \frac{1}{5}\left(\frac{e^t\left(1 - e^{Nt}\right)}{1 - e^t}\right)
\end{aligned}
$$

2- $X \sim Bernoulli(0.7)$

$$f(x) = \begin{cases} 0.7^x 0.3^{1-x} & x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$M_X(t) = 0.7e^t + 0.3$$

3- $Y \sim Binomial(5, 0.8)$

$$f(x) = \begin{cases} \binom{5}{y} 0.8^x 0.2^{5-y} & y = 0, 1, ..., 5 \\ 0 & \text{otherwise} \end{cases}$$

$$M_X(t) = \left(0.8e^t + 0.2\right)^5$$