Poisson distribution: A random variable X is said to have a Poisson distribution, denoted by $X \sim Poisson(p)$, if its PMF has the following form:

$$f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{for } x = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is the expected rate of occurrence of a certain event. This probability distribution models the behavior of a random variable X that represents the number of times a certain event occurs over a specified period. For instance, the number of monthly car accidents in a certain city, the number of daily COVID cases in Iraq, and the number of defective items per container produced by a specific machine.

Mean, variance, MGF, and CDF: by definition of the mean, variance, and MGF are defined as follows:

$$\begin{aligned} E[X] &= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \end{aligned}$$

let $k = x - 1$, then $k = 0, 1, 2, ...$

$$E[X] = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$
$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

to find $E[X^2]$, we obtain $E[X(X-1)]$ first. Then calculate $E[X^2] = E[X(X-1)] + E[X]$

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \sum_{x=2}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!}, \text{ when } x = 0 \text{ the first two terms are zeros} \\ &= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-1)!} \end{aligned}$$

let $k = x - 2$, then $k = 0, 1, 2, ...$

$$E[X(X-1)] = \lambda^2 e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$
$$= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

$$E[X^2] = E[X(X-1)] + E[X]$$
$$= \lambda^2 + \lambda$$

$$\text{Var}(X) = E[X^2] - E[X]^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

notice that the mean and variance of Poisson random variable are equal.

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)} \end{aligned}$$

Example 3.1: Let X be a random variable that follows a Poisson distribution with rate λ. Find the value of λ, such that $P(X=0) = P(X=1)$. Then find $E[X]$, $\text{Var}(X)$, and $M_X(t)$.

Solution: To find the value of λ, we have:

$$\begin{aligned} P(X=0) &= P(X=1) \\ \implies \frac{\lambda^0 e^{-\lambda}}{0!} &= \frac{\lambda^1 e^{-\lambda}}{1!} \\ \implies e^{-\lambda} &= \lambda e^{-\lambda} \\ \implies \lambda &= \frac{e^{-\lambda}}{e^{-\lambda}} = 1 \end{aligned}$$

hence the PMF of X is

$$f(x) = \begin{cases} \frac{e^{-1}}{x!} & \text{for } x = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

therefore $E[X] = \text{Var}(X) = 1$, and $M_X(t) = e^{e^t-1}$.

Example 3.2: Let $X_1 \sim Poisson(4)$, $X_2 \sim Poisson(6)$, and $X_3 \sim$

$Poisson(8)$ are three independent random variables. Do the following:

1- Find the mean and variance of $Y_1 = X_1 + X_2$, $Y_2 = X_1 + X_3$, and $Y_3 = X_1 + X_2 + X_3$.

2- The MGF of $Y_3 = X_1 + X_2 + X_3$.

3- Find $P(Y_1 \geq 2)$.

Solution: 1- Find the mean and variance

$$E[Y_1] = E[X_1 + X_2]$$
$$= E[X_1] + E[X_2] = 4 + 6 = 10$$

$$E[Y_2] = E[X_1 + X_3]$$
$$= E[X_1] + E[X_3] = 4 + 8 = 12$$

$$E[Y_3] = E[X_1 + X_2 + X_3]$$
$$= E[X_1] + E[X_2] + E[X_3] = 4 + 6 + 8 = 18$$

$$\mathrm{Var}(Y_1) = \mathrm{Var}(X_1 + X_2)$$
$$= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 4 + 6 = 10$$

$$\begin{aligned}\mathrm{Var}(Y_2) &= \mathrm{Var}(X_1 + X_2)\\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_3) = 4 + 8 = 12\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Y_3) &= \mathrm{Var}(X_1 + X_2 + X_3)\\ &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = 4 + 6 + 8 = 18\end{aligned}$$

we will see later this semester that $Y_1 \sim Poisson(\lambda_1+\lambda_2)$, $Y_2 \sim Poisson(\lambda_1+\lambda_3)$, and $Y_3 \sim Poisson(\lambda_1 + \lambda_2 + \lambda_3)$.

2- Find MGF of $Y_3 = X_1 + X_2 + X_3$

$$\begin{aligned}M_{Y_3}(t) = E\left[e^{tY_3}\right] &= E\left[e^{t(X_1+X_2+X_3)}\right]\\ &= E\left[e^{tX_1}e^{tX_2}e^{tX_3}\right]\\ &= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] E\left[e^{tX_3}\right]\\ &= M_{X_1}(t)M_{X_2}(t)M_{X_3}(t)\\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}e^{\lambda_3(e^t-1)}\\ &= e^{4(e^t-1)}e^{6(e^t-1)}e^{8(e^t-1)}\\ &= e^{(4+6+8)(e^t-1)} = e^{18(e^t-1)}\end{aligned}$$

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\begin{aligned}
P(Y_1 \geq 2) &= 1 - P(Y_1 \leq 1) \\
&= 1 - P(X_1 + X_2 \leq 1) \\
&= 1 - (P(X_1 = 0, X_1 = 0) + P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0)) \\
&= 1 - \left(\frac{4^0 e^{-4}}{0!} \times \frac{6^0 e^{-6}}{0!} + \frac{4^0 e^{-4}}{0!} \times \frac{6^1 e^{-6}}{1!} + \frac{4^1 e^{-4}}{1!} \times \frac{6^0 e^{-6}}{0!} \right) \\
&= 1 - \left(e^{-10} + 6e^{-10} + 4e^{-10} \right) \\
&= 1 - 11e^{-10} = 0.9995
\end{aligned}
$$

Now assume that $Y_1 \sim Poisson(10)$, find $P(Y_1 \geq 2)$. (**Homework**)

Poisson distribution as an approximation to binomial distribution

Let X follows a binomial distribution with the total number of trials n and the probability of success p. If n is very large (say $n \geq 100$), and p is small, such that $np \leq 10$, then the distribution of X can be approximated using the Poisson distribution with rate $\lambda = np$. (**Proof is Homework**)

Example 3.3: Suppose that 0.003 of bolts made by a machine are defective, the defectives occurring at random during production. If the bolts are packaged in boxes of 100, what is the Poisson approximation that a given box will contain x defectives?

Solution: Let X represents the number of defective bolts in each box. Then X has a binomial distribution with $n = 100$ and $p = 0.003$. Now, since

$$np = 100(0.003) = 0.3 \leq 10$$

we can use Poisson distribution with rate $\lambda = np = 0.3$ to approximate the distribution of X. That is:

$$\begin{aligned} f(x) &\approx \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{0.3^x e^{-0.3}}{x!} \end{aligned}$$

for $x = 0, 1, 2, ...$

Example 3.4 (Homework): Given that 0.0004 of vehicles break down when driving through a certain tunnel find the probability of (a) no car breaks down (b) at least two cars break down in an hour when 2,000 vehicles enter the tunnel.

Example 3.5: Let $X \sim Poisson(\lambda)$, and assume that the conditional distribution of Y given $X = x$ is $Binomial(x, p)$. Show that $Y \sim Poisson(p\lambda)$.

Solution: We have

$$f_X(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & x = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \binom{x}{y} p^y (1-p)^{x-y} & y = 0, 1, ..., x;\ \text{and } 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

to find the marginal distribution of Y, we first need to find the joint distribution of X and Y. By definition, the conditional PMF can be written as

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ \implies f_{X,Y}(x,y) &= f_{Y|X}(y|x) f_X(x) \\ &= \binom{x}{y} p^y (1-p)^{x-y} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{p^y e^{-\lambda}}{y!} \times \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} \end{aligned}$$

for $y = 0, 1, ..., x$; $x \ge y$; $0 \le p \le 1$; and $\lambda > 0$.

Now to find the marginal PMF of Y we need to sum over x. That is:

$$\begin{aligned} f_Y(y) &= \sum_{x=y}^{\infty} \frac{p^y e^{-\lambda}}{y!} \times \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} \\ &= \frac{p^y e^{-\lambda}}{y!} \left(\sum_{x=y}^{\infty} \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} \right) \end{aligned}$$

let $k = x - y$, then $x = k + y$, and $k = 0, 1, 2, ...$

$$\begin{aligned} f_Y(y) &= \frac{p^y e^{-\lambda}}{y!} \left(\sum_{k=0}^{\infty} \frac{\lambda^{k+y}(1-p)^k}{k!} \right) \\ &= \frac{p^y e^{-\lambda} \lambda^y}{y!} \left(\sum_{k=0}^{\infty} \frac{((1-p)\lambda)^k}{k!} \right) \\ &= \frac{(p\lambda)^y e^{-\lambda} e^{(1-p)\lambda}}{y!} \\ &= \frac{(p\lambda)^y e^{-\lambda} e^{\lambda} e^{-p\lambda}}{y!} = \frac{(p\lambda)^y e^{-p\lambda}}{y!} \end{aligned}$$

the later is the PMF of Poisson distribution with rate parameter $p\lambda$. Therefore, $Y \sim Poisson(p\lambda)$.