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The continuous uniform distribution: A continuous random variable X is said to have a continuous uniform distribution, denoted by $X \sim Uniform(a, b)$, if its PDF has the following form:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are the parameters of the distribution. If $a = 0$ and $b = 1$, then the distribution is called a *standard uniform distribution*.

Mean, variance, MGF, and CDF: By definition of the mean, variance, MGF, and the CDF are defined as follows:

$$\begin{aligned} E[X] &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \int_a^b x dx \\ &= \frac{1}{b-a} \left(\frac{x^2}{2} \Big|_a^b \right) \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} \end{aligned}$$

$$
\begin{aligned}
E[X^2] &= \frac{1}{b-a}\int_a^b x^2 dx \\
&= \frac{1}{b-a}\left(\left.\frac{x^3}{3}\right|_a^b\right) \\
&= \frac{b^3-a^3}{3(b-a)} \\
&= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 \\
&= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4} \\
&= \frac{4(b^2+ab+a^2)-3(b^2+2ab+a^2)}{12} \\
&= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}
$$

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \frac{1}{b-a}\int_a^b e^{tx} dx \\
&= \frac{1}{b-a}\left(\left.\frac{1}{t}e^{tx}\right|_a^b\right) = \frac{e^{tb}-e^{ta}}{t(b-a)}
\end{aligned}
$$

for $t \neq 0$, and $M_X(t) = 1$ for $t = 0$ **(prove it)**.

$$
\begin{aligned}
F(x) &= P(X \le x) \\
&= \int_a^x \frac{1}{b-a} du \\
&= \frac{1}{b-a} \left(u|_a^x \right) = \frac{x-a}{b-a}
\end{aligned}
$$

$$
F(x) = \begin{cases} \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x \ge b \\ 0 & \text{for } x < a \end{cases}
$$

Example 4.1: Let X be a random variable that follows a uniform distribution over the interval $[a, b]$. Find the median of X.

Solution: The median (m) satisfies the following

$$
\begin{aligned}
P(X \le m) &= \frac{1}{2} \\
\implies F(m) &= \frac{1}{2} \\
\implies \frac{m-a}{b-a} &= \frac{1}{2} \\
\implies m &= \frac{b-a}{2} + a \\
\implies m &= \frac{b+a}{2}
\end{aligned}
$$

hence the mean and median of uniform distribution are equal.

The exponential distribution: A continuous random variable X is said to be exponentially distributed, denoted by $X \sim Expo(\lambda)$, if its PDF has the following form:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is the *rate parameter*. The exponential distribution is often used to describe the behavior of a random variable that represents the time between events in a Poisson point process.

Mean, variance, MGF, and CDF: By definition of the mean, the variance, the MGF, and the CDF are defined as follows:

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} dx$$

we evaluate the integral by

$$\int udv = uv - \int vdu$$

let $u = x$, then $du = dx$, and let $dv = \lambda e^{-\lambda x} dx$, then $v = -e^{-\lambda x}$. Hence:

$$\begin{aligned}
\int_0^\infty x\lambda e^{-\lambda x} dx &= x e^{-\lambda x}\Big|_0^\infty - \int_0^\infty (-e^{-\lambda x}) dx \\
&= \underbrace{\left(\infty e^{-\lambda(\infty)} - 0 e^{-\lambda(0)}\right)}_{zero} - \left(\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty\right) \\
&= -\left(\frac{1}{\lambda}\left(e^{-\lambda(\infty)} - e^{-\lambda(0)}\right)\right) \\
&= -\left(\frac{1}{\lambda}(0-1)\right) = \frac{1}{\lambda}
\end{aligned}$$

similarly,

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\theta x} dx = \frac{2}{\lambda^2}, \quad (\textbf{Homework})$$

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}
\end{aligned}$$

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx \\
&= \int_0^\infty \lambda e^{-(\lambda-t)x}dx \\
&= -\frac{\lambda}{\lambda-t}e^{-\lambda x}\Big|_0^\infty \\
&= -\frac{\lambda}{\lambda-t}\left(e^{-\lambda(\infty)}-e^{-\lambda(0)}\right) \\
&= -\frac{\lambda}{\lambda-t}(0-1) = \frac{\lambda}{\lambda-t}
\end{aligned}
$$

for $t < \lambda$.

$$
\begin{aligned}
F(x) &= P(X \le x) \\
&= \int_0^x \lambda e^{-\lambda u}du \\
&= -e^{-\lambda u}\Big|_0^x \\
&= -\left(e^{-\lambda x}-e^{-\lambda(0)}\right) = 1-e^{-\lambda x}
\end{aligned}
$$

$$
F(x) = \begin{cases} 1-e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}
$$

Alternative parameterization: The exponential distribution is sometimes parameterized in terms of the scale parameter $\beta = 1/\lambda$. In this case, the PDF, the mean, the variance, the MGF, and the CDF are defined as follows:

$$f(x) = \begin{cases} \frac{1}{\beta}e^{-x/\beta} & \text{for } x \geq 0, \text{ and } \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \beta$$

$$\text{Var}(X) = \beta^2$$

$$M_X(t) = \frac{1}{1-\beta t}$$

$$F(x) = \begin{cases} 1 - e^{-x/\beta} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.2: On the average, a certain computer part lasts 10 years. The length of time the computer part lasts is exponentially distributed.

a) What is the probability that a computer part lasts more than 7 years?

b) On the average, how long would five computer parts last if they are used one after another?

c) What is the probability that a computer part lasts between nine and 11 years?

Solution: Let X represents the time (in years) a computer part lasts. We have the average time is $E[X] = 10$. But $E[X] = \frac{1}{\lambda}$. Therefore $\lambda = \frac{1}{E[X]} = \frac{1}{10} = 0.1$

Hence the distribution of X is

$$f(x) = \begin{cases} 0.1e^{-0.1x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) The probability that a computer part lasts more than 7 years

$$\begin{aligned} P(X > 7) &= \int_7^{\infty} 0.1e^{-0.1x} dx \\ &= -\left(e^{-0.1(\infty)} - e^{-0.1(7)}\right) \\ &= e^{-0.7} = 0.4966 \end{aligned}$$

b) On the average, each computer part lasts 10 years. Therefore, five computer parts would last $5 \times 10 = 50$ years when they are used one after another.

c) The probability that a computer part lasts between 9 and 11 years is

$$\begin{aligned} P(9 < X < 11) &= \int_9^{11} 0.1e^{-0.1x} dx \\ &= -\left(e^{-0.1(11)} - e^{-0.1(9)}\right) \\ &= e^{-0.9} - e^{-1.1} = 0.0737 \end{aligned}$$

Example 4.3: Let X be an exponentially distributed random variable with rate parameter λ. Find the median of X.

Solution: The median (m) satisfies the following

$$\begin{aligned} P(X \leq m) &= \frac{1}{2} \\ \implies F(m) &= \frac{1}{2} \\ \implies 1 - e^{-\lambda m} &= \frac{1}{2} \\ \implies e^{-\lambda m} &= \frac{1}{2} \\ \implies -\lambda m &= \ln\left(\frac{1}{2}\right) \\ \implies m &= -\frac{\ln\left(\frac{1}{2}\right)}{\lambda} \\ \implies m &= -\frac{\ln(1) - \ln(2)}{\lambda} \\ \implies m &= \frac{\ln(2)}{\lambda} \end{aligned}$$

Example 4.4 (Homework): Let X be an exponentially distributed random variable with rate parameter λ. Find Q_1 and Q_3 of X, where $Q_1$and Q_3 are the first and third quartiles, respectively.