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The gamma distribution: A continuous random variable X is said to have a *gamma distribution*, denoted by $X \sim Gamma(\alpha, \beta)$, if its PDF has the following form:

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is shape parameter, and $\beta > 0$ is the rate parameter, and $\Gamma(\alpha)$ is called the *gamma function* and is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$$

Notice that $\Gamma(\alpha) = (\alpha - 1)! = (\alpha - 1)(\alpha - 2)! = (\alpha - 1)\Gamma(\alpha - 1)$, and $\Gamma(1/2) = \sqrt{\pi}$. Also if $\alpha = 1$, then the gamma distribution reduces to the exponential distribution with rate parameter β; i.e. $Gamma(1, \beta) \equiv Expo(\beta)$. That is

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \Big|_{\alpha=1} = \beta e^{-\beta x}$$

The gamma distribution is frequently used to model waiting times. For instance, in survival studies, the survival time before a certain event happens is a random variable that is usually modeled with a gamma distribution.

Mean, variance, and MGF: By definition of the mean, variance, and the MGF are defined as follows:

$$\begin{aligned} E[X] &= \int_0^\infty x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} dx \end{aligned}$$

notice that $x^\alpha e^{-\beta x}$ is the kernel of a $Gamma(\alpha + 1, \beta)$ distribution with a PDF $f(x) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\beta x}$. Where the *kernel of the distribution* is the part of the probability function that contains x. Now recall that $\int_0^\infty f(x)dx = 1$. Therefore

$$\begin{aligned} \int_0^\infty \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\beta x} dx = 1 \\ \implies \int_0^\infty x^\alpha e^{-\beta x} dx = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \end{aligned}$$

hence

$$\begin{aligned} E[X] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \right) \\ &= \left(\frac{\alpha!}{(\alpha-1)!} \right) \left(\frac{1}{\beta} \right) = \frac{\alpha}{\beta} \end{aligned}$$

similarly

$$
\begin{aligned}
E[X^2] &= \int_0^\infty x^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x} dx
\end{aligned}
$$

notice that $x^{\alpha+1}e^{-\beta x}$ is the kernel of $Gamma(\alpha + 2, \beta)$.

$$
\begin{aligned}
E[X^2] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}} \right) \\
&= \left(\frac{(\alpha+1)!}{(\alpha-1)!} \right) \left(\frac{1}{\beta^2} \right) = \frac{\alpha(\alpha+1)}{\beta^2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta} \right)^2 = \frac{\alpha}{\beta^2}
\end{aligned}
$$

$$
\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx
\end{aligned}
$$

notice that $x^{\alpha-1}e^{-(\beta-t)x}$ is the kernel of $Gamma(\alpha, \beta - t)$

$$
M_X(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{(\beta-t)^\alpha} \right) = \left(\frac{\beta}{\beta - t} \right)^\alpha
$$

Alternative parameterization: The gamma distribution is sometimes parameterized in terms of the scale parameter instead of the rate parameter.

In this case, the PDF, the mean, the variance, the MGF, and the CDF are defined as follows:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} & \text{for } x \geq 0, \text{ and } \alpha, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \alpha\beta$$
$$\text{Var}(X) = \alpha\beta^2$$
$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$$

Example 5.1: Let X be a random variable that follows $Gamma(\alpha, \beta)$ distribution . Find the mode and the harmonic mean of X.

Solution: To find the mode, we need to solve the following equation with respect to x:

$$\frac{d}{dx} f(x) := 0$$

in order to find the derivative of $f(x)$ with respect to x, we need to take the natural log of $f(x)$, then solve

$$\frac{d}{dx} \ln f(x) := 0$$

that is

$$\ln f(x) = \ln\left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\right)$$
$$= \alpha \ln \beta - \ln \Gamma(\alpha) + (\alpha - 1)\ln x - \beta x$$

now to find the mode

$$\frac{d}{dx} \ln f(x) = 0 - 0 + \frac{\alpha - 1}{x} - \beta := 0$$
$$\implies \frac{\alpha - 1}{x} = \beta$$
$$\implies x = \frac{\alpha - 1}{\beta}$$

to verify that the mode of X is $\frac{\alpha-1}{\beta}$, we need to show that the second derivative is negative when $x = \frac{\alpha-1}{\beta}$. That is:

$$\frac{d^2}{dx^2} \ln f(x) = -\frac{\alpha - 1}{x^2}$$

put $x = \frac{\alpha-1}{\beta}$, we get

$$\frac{d^2}{dx^2} \ln f(x) = -\frac{\alpha - 1}{\left(\frac{\alpha-1}{\beta}\right)^2}$$
$$= -\frac{\beta}{\alpha - 1}$$

the second derivative is negative when $\alpha > 1$. Hence, the mode of gamma distribution is $x = \frac{\alpha-1}{\beta}$, and it only exist when $\alpha > 1$.

To find the harmonic mean

$$\bar{H} = \frac{1}{E\left[\frac{1}{X}\right]}$$

$$\begin{aligned} E\left[\frac{1}{X}\right] &= \int_0^\infty \left(\frac{1}{x}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-2} e^{-\beta x} dx \end{aligned}$$

notice that $x^{\alpha-2}e^{-\beta x}$ is the kernel of $Gamma(\alpha - 1, \beta)$.

$$\begin{aligned} E\left[\frac{1}{X}\right] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha-1)}{\beta^{\alpha-1}}\right) \\ &= \beta\left(\frac{(\alpha-2)!}{(\alpha-1)!}\right) = \frac{\beta}{\alpha-1} \end{aligned}$$

$$\begin{aligned} \bar{H} &= \frac{1}{E\left[\frac{1}{X}\right]} \\ &= \frac{\alpha-1}{\beta} \end{aligned}$$

Thus, the mode and the harmonic mean of gamma distribution are equal.