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Steps of the change-of-variable approach:

1. Show that $Y = g(X)$ is a monotone function. The function $g(X)$ is increasing if $\frac{d}{dx}g(x) > 0$, and is decreasing if $\frac{d}{dx}g(x) < 0$.
2. Find the support of Y.
3. Solve for the inverse of the transformation; i.e., find $x = g^{-1}(y)$.
4. Obtain $\frac{dx}{dy}$.
5. The pdf of Y is $f_Y(y) = f_X(g^{-1}(y)) \mid \frac{dx}{dy} \mid$.

Example 8.1: Let X be a random variable with the following PDF

$$f_X(x) = \begin{cases} 4x^3 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

consider random variable $Y = -\ln X$. Use the change-of-variable approach to find the distribution of Y.

Solution: Here are the steps of the above algorithm:

1. We have $\frac{d}{dx}g(x) = -\frac{1}{x} < 0$ because x is positive. Therefore, $g(\cdot)$ is a decreasing function.
2. The support of $Y = -\ln X$ is $(0,\infty)$.
3. If $y = -\ln x$ then $x = e^{-y}$.
4. $\mid \frac{dx}{dy} \mid = \mid -e^{-y} \mid = e^{-y}$.

5. Thus the PDF of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \mid \frac{dx}{dy} \mid \\ &= 4\left(e^{-y}\right)^3 e^{-y} = 4e^{-4y} \end{aligned}$$

hence $Y \sim Expo(4)$.

Example 8.2 (Homework): Let X be a random variable with the following PDF

$$f_X(x) = \begin{cases} 2xe^{-x^2} & \text{for } 0 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

consider random variable $Y = X^2$. Use the change-of-variable approach to find the PDF of Y.

3- The moment generating function method: This method is mainly used to find the distribution of the sum of independent random variables. That is, if $X_1, X_2, ..., X_k$ are k independent random variables, then we can use the *moment generating function method* to find the distribution of $Y = \sum_{j=1}^{k} X_j$. Notice that if $X_1, X_2, ..., X_k$ are k independent random variables, then

$$\begin{aligned} M_{(Y)}(t) &= E\left[e^{tY}\right] \\ &= E\left[e^{t(X_1+X_2+\cdots+X_k)}\right] \\ &= E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] \cdots E\left[e^{tX_k}\right] \qquad \text{by independence} \\ &= M_{(X_1)}(t) M_{(X_2)}(t) \cdots M_{(X_k)}(t) \end{aligned}$$

Therefore, if we know the MGFs of $X_1, X_2, ..., X_k$, we can find the MGF of $Y = \sum_{j=1}^{k} X_j$, and hence the distribution of Y.

Example 8.3 (Homework): (*Additive property of Bernoulli distribution*) Let $X_1, X_2, ..., X_n$ be n independent Bernoulli trials. Use the MGF method to find the distribution of $Y = \sum_{j=1}^{n} X_j$.

Example 8.4: (*Additive property of Binomial distribution*)

Let $Y_1 \sim Binomial(n_1, p)$, and $Y_2 \sim Binomial(n_2, p)$ be two independent random variables. Use the MGF method to show that $Y \sim Binomial(n, p)$, where $n = n_1 + n_2$.

Solution: Since $Y_1 \sim Binomial(n_1, p)$, and $Y_2 \sim Binomial(n_2, p)$, then

$$M_{Y_1}(t) = \left(pe^t + (1-p)\right)^{n_1} \qquad \text{and} \qquad M_{Y_2}(t) = \left(pe^t + (1-p)\right)^{n_2}$$

Further, since Y_1 and Y_2 are independent, we have

$$\begin{aligned} M_{(Y)}(t) &= M_{Y_1}(t) M_{Y_2}(t) \\ &= \left(pe^t + (1-p)\right)^{n_1} \left(pe^t + (1-p)\right)^{n_2} \\ &= \left(pe^t + (1-p)\right)^{n_1+n_2} = \left(pe^t + (1-p)\right)^{n} \end{aligned}$$

that is, $Y \sim Binomial(n, p)$. Hence the density function of $Y = Y_1 + Y_2$ is given by

$$f_Y(y) = \begin{cases} \binom{n}{y} p^y (1-p)^{n-y} & \text{for } y = 0, 1, ..., n \\ 0 & \text{otherwise} \end{cases}$$

Example 8.5 (Homework): Let $Y_1 \sim Binomial(3, 0.6)$, $Y_2 \sim Binomial(8, 0.6)$, and $Y_3 \sim Binomial(6, 0.6)$ be three independent random variables. Use the MGF method to find the distribution of $Y = \sum_{j=1}^{3} Y_j$.

Example 8.6: (*Additive property of Poisson distribution*) Let $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$ be two independent random variables. Use the MGF method to find the distribution of $Y = X_1 + X_2$.

Solution: Since $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$, then

$$M_{X_1}(t) = e^{\lambda_1(e^t-1)} \qquad \text{and} \qquad M_{X_2}(t) = e^{\lambda_2(e^t-1)}$$

Further, since X_1 and X_2 are independent, we have

$$\begin{aligned} M_{(Y)}(t) &= M_{X_1}(t)M_{X_2}(t) \\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} \\ &= e^{\lambda_1(e^t-1)+\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_1)(e^t-1)} \end{aligned}$$

that is, $Y \sim Poisson(\lambda_1 + \lambda_2)$. Hence the density function of $Y = X_1 + X_2$ is given by

$$f_Y(y) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^y}{y!} & \text{for } y = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

Example 8.7 (Homework): Let $X_1, X_2,$ and X_3 be three independent random variables from Poisson distribution with $\lambda_j = j$, for $j = 1, 2, 3$. Find the distribution of (1) $W = X_1 + X_2$, (2) $Y = X_1 + X_2 + X_3$, and (3) $Z = X_2 + X_3$.

Example 8.8: Let $X_1, X_2, ..., X_k$ be k independent random variables from $N(\mu_j, \sigma_j^2)$, for $j = 1, 2, ..., k$. Find the distribution of $Y = \sum_{j=1}^{k} X_j$.

Solution: Since $X_j \sim N(\mu_j, \sigma_j^2)$, then

$$M_{X_j}(t) = e^{t\mu_j + \frac{1}{2}t^2\sigma_j^2}$$

Further, since $X_1, X_2, ..., X_k$ are independent, we have

$$\begin{aligned}
M_{(Y)}(t) &= M_{X_1}(t)M_{X_1}(t)\cdots M_{X_k}(t) \\
&= \prod_{j=1}^{k}\left(e^{t\mu_j+\frac{1}{2}t^2\sigma_j^2}\right) \\
&= \left(e^{t\mu_1+\frac{1}{2}t^2\sigma_1^2}\right)\left(e^{t\mu_2+\frac{1}{2}t^2\sigma_2^2}\right)\cdots\left(e^{t\mu_k+\frac{1}{2}t^2\sigma_k^2}\right) \\
&= e^{t\mu_1+\frac{1}{2}t^2\sigma_1^2+t\mu_2+\frac{1}{2}t^2\sigma_2^2+\cdots+t\mu_k+\frac{1}{2}t^2\sigma_k^2} \\
&= e^{t\sum_{j=1}^{k}\mu_j+\frac{1}{2}t^2\sum_{j=1}^{k}\sigma_j^2}
\end{aligned}$$

that is, $Y \sim N(\mu_Y, \sigma_Y^2)$, where $\mu_Y = \sum_{j=1}^{k}\mu_j$, and $\sigma_Y^2 = \sum_{j=1}^{k}\sigma_j^2$. Hence the density function of $Y = \sum_{j=1}^{k} X_j$ is given by

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{1}{2}\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2}$$

Properties of the linear transformation of normal random variables:

1. If $X \sim N(\mu, \sigma^2)$, then $Y = aX + b$ follows a $N(\mu_Y, \sigma_Y^2)$, where $\mu_Y = a\mu + b$, and $\sigma_Y^2 = a^2\sigma^2$.

2. If $X_1, X_2, ..., X_n$ are n independent random variables with $X_j \sim N(\mu_j, \sigma_j^2)$, for $j = 1, 2, ..., n$. Then $Y = \sum_{j=1}^{n} a_jX_j$ has a $N(\mu_Y, \sigma_Y^2)$, where $\mu_Y = \sum_{j=1}^{n} a_j\mu_j$, and $\sigma_Y^2 = \sum_{j=1}^{n} a_j^2\sigma_j^2$.

Example 8.9 (Homework): Let $X_1, X_2, ..., X_k$ be k independent random variables from $N(\mu, \sigma^2)$. Find the distribution of $Y = \sum_{j=1}^{k} X_j$.

Example 8.10: Let $X_1, X_2, ..., X_n$ be n independent random variables from $N(\mu, \sigma^2)$. Show that the distribution of $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ is $N(\mu, \frac{\sigma^2}{n})$.

Solution: Since $X_j \sim N(\mu, \sigma^2)$, then

$$M_{X_j}(t) = e^{t\mu+\frac{1}{2}t^2\sigma^2}, \qquad \text{for } j = 1, 2, ..., n$$

Further, since $X_1, X_2, ..., X_n$ are independent, we have

$$\begin{aligned}
M_{(\bar{X})}(t) &= E\left[e^{t\bar{X}}\right] \\
&= E\left[e^{t\left(\frac{\sum_{j=1}^{n} X_j}{n}\right)}\right] \\
&= E\left[e^{\frac{t}{n}(X_1+X_2+\cdots+X_n)}\right] \\
&= E\left[e^{\frac{t}{n}X_1}\right] E\left[e^{\frac{t}{n}X_2}\right] \cdots E\left[e^{\frac{t}{n}X_n}\right] \\
&= M_{(X_1)}(t/n) M_{(X_2)}(t/n) \cdots M_{(X_n)}(t/n) \\
&= \left(e^{\frac{t}{n}\mu+\frac{1}{2}\frac{t^2}{n^2}\sigma^2}\right)\left(e^{\frac{t}{n}\mu+\frac{1}{2}\frac{t^2}{n^2}\sigma^2}\right)\cdots\left(e^{\frac{t}{n}\mu+\frac{1}{2}\frac{t^2}{n^2}\sigma^2}\right) \\
&= \left(e^{\frac{t}{n}\mu+\frac{1}{2}t^2\frac{\sigma^2}{n^2}}\right)^n = e^{t\mu+\frac{1}{2}t^2\frac{\sigma^2}{n}}
\end{aligned}$$

hence $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. The PDF of $\bar{X}$ is

$$\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{2\pi}\frac{\sigma}{\sqrt{n}}} e^{-\frac{1}{2}\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2} \\
&= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-\frac{n}{2\sigma^2}\left(\bar{X}-\mu\right)^2}
\end{aligned}$$

Example 8.11 (Homework): Let $X \sim N(\mu, \sigma^2)$. Use the change-of-variable technique and the MGF method to find the distribution of $Y = aX + b$, where a and b are real numbers.

Example 8.12 (Homework): Let $X \sim N(\mu, \sigma^2)$. Use the MGF method to find the distribution of $Z = \frac{X-\mu}{\sigma}$.

Example 8.13: Let $X \sim Uniform(0, 1)$. Find the distribution of $Y = -\frac{\ln X}{2}$ using the MGF method.

Solution:

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E\left[e^{t\left(-\frac{\ln X}{2}\right)}\right] \\
&= E\left[e^{\left(-\frac{t}{2}\right)(\ln X)}\right] \\
&= E\left[\left(e^{-\ln X}\right)^{-\frac{t}{2}}\right] \\
&= E\left[X^{-\frac{t}{2}}\right] \\
&= \int_0^1 X^{-\frac{t}{2}} dx \\
&= \left.\frac{X^{-\frac{t}{2}+1}}{-\frac{t}{2}+1}\right|_0^1 \\
&= \frac{1}{1-\frac{t}{2}} = \frac{2}{2-t}
\end{aligned}
$$

that is, $Y \sim Expo(2)$.

Example 8.14: (*Additive property of Exponential distribution*) Let $X_j \sim Expo(\lambda)$, for $j = 1, 2, ..., k$, are k independent random variables. Find the distribution of $Y = \sum_{j=1}^{k} X_j$ using the MGF method.

Solution:

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E\left[e^{t(X_1+X_2+\cdots+X_k)}\right] \\
&= E\left[e^{tX_1}e^{tX_2}\cdots e^{tX_k}\right] \\
&= E\left[e^{tX_1}\right]E\left[e^{tX_2}\right]\cdots E\left[e^{tX_k}\right] \\
&= \left(\frac{\lambda}{\lambda-t}\right)\left(\frac{\lambda}{\lambda-t}\right)\cdots\left(\frac{\lambda}{\lambda-t}\right) \\
&= \left(\frac{\lambda}{\lambda-t}\right)^k
\end{aligned}
$$

that is, $Y \sim Gamma(k, \lambda)$.

Example 8.15: (*Additive property of Gamma distribution*) Let $X_j \sim Gamma(\alpha, \beta)$, for $j = 1, 2, ..., k$, are k independent random variables. Find the distribution of $Y = \sum_{j=1}^{k} X_j$ using the MGF method.

Solution:

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E\left[e^{t(X_1+X_2+\cdots+X_k)}\right] \\
&= E\left[e^{tX_1}e^{tX_2}\cdots e^{tX_k}\right] \\
&= E\left[e^{tX_1}\right]E\left[e^{tX_2}\right]\cdots E\left[e^{tX_k}\right] \\
&= \left(\frac{\beta}{\beta - t}\right)^{\alpha}\left(\frac{\beta}{\beta - t}\right)^{\alpha}\cdots\left(\frac{\beta}{\beta - t}\right)^{\alpha} \\
&= \left(\frac{\beta}{\beta - t}\right)^{k\alpha}
\end{aligned}
$$

that is, $Y \sim Gamma(k\alpha, \beta)$.