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Bivariate Transformations:

Theorem 9.1: Let X and Y be two continuous random variables with joint density $f_{X,Y}(x, y)$. Let $U = g_1(X, Y)$ and $V = g_2(X, Y)$ be functions of X and Y. If $g_1(x, y)$ and $g_2(x, y)$ are invertible functions such that x and y can be written as $x = h_1(u, v)$ and $y = h_2(u, v)$, then the joint density of U and V is given by:

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))\,|J|$$

where

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Steps of the bivariate transformation approach:

1. Find the support of U and V.
2. Solve for the inverse of the transformations; i.e., find $x = h_1(u, v)$ and $y = h_2(u, v)$.
3. Obtain $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, and $\frac{\partial y}{\partial v}$.
4. Calculate

$$J = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

5. The joint PDF of U and V is

$$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))\,|J|$$

Example 9.1: Let X and Y be two independent random variables both follows $Gamma(\alpha, \beta)$. If $U = \frac{X}{X+Y}$, $V = X + Y$, find the joint PDF of U and V. Then find the marginal PDF of U.

Solution: Since X and Y are independent random variables both from $Gamma(\alpha, \beta)$, then the joint PDF of X and Y is

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) f_Y(y) \\ &= \left(\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}\right) \\ &= \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)} (xy)^{\alpha-1} e^{-\beta(x+y)} \end{aligned}$$

for $x, y \geq 0$, and $\alpha, \beta > 0$.

Now we need to find the support of U and V. We have $v = x + y$, and $x, y \geq 0$, therefore $v \geq 0$. We also have $u = \frac{x}{x+y}$, and $x, y \geq 0$, therefore $0 \leq u \leq 1$

Now we need to find the inverse of U and V. That is

$$\begin{aligned} v = x + y &\implies y = v - x \text{ .....(1)} \\ u = \frac{x}{x+y} &\implies x = uv \text{ .....(2)} \end{aligned}$$

substitute 2 in 1, we get

$$y = v - uv$$

Thus $\frac{\partial x}{\partial u} = v$, $\frac{\partial x}{\partial v} = u$, $\frac{\partial y}{\partial u} = -v$, and $\frac{\partial y}{\partial v} = 1 - u$. Therefore

$$
\begin{aligned}
J &= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \\
&= \det \begin{pmatrix} v & u \\ \\ -v & 1-u \end{pmatrix} \\
&= |v(1-u) - (u(-v))| \\
&= |v - uv - uv| = v
\end{aligned}
$$

therefor the joint PDF of U and V is

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(u,v)\,|J| \\
&= \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}(uv(v-uv))^{\alpha-1}e^{-\beta(uv+v-uv)}(v) \\
&= \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}v^{2\alpha-1}(u(1-u))^{\alpha-1}e^{-\beta v}
\end{aligned}
$$

Hence

$$
f_{U,V}(u,v) = \begin{cases} \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}v^{2\alpha-1}(u(1-u))^{\alpha-1}e^{-\beta v} & \text{for } 0 \le u \le 1, \text{ and } v \ge 0 \\ \\ 0 & \text{otherwise} \end{cases}
$$

to find the marginal PDF of U

$$
\begin{aligned}
f_U(u) &= \int_0^\infty \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}v^{2\alpha-1}(u(1-u))^{\alpha-1}e^{-\beta v}dv \\
&= \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}(u(1-u))^{\alpha-1}\int_0^\infty v^{2\alpha-1}e^{-\beta v}dv
\end{aligned}
$$

Notice that $v^{2\alpha-1}e^{-\beta v}$ is the kernel of $Gamma(2\alpha, \beta)$ distribution. Therefore

$$\int_0^\infty v^{2\alpha-1}e^{-\beta v}dv = \frac{\Gamma(2\alpha)}{\beta^{2\alpha}}$$

so that

$$\begin{aligned} f_U(u) &= \frac{\beta^{2\alpha}}{\Gamma(\alpha)\Gamma(\alpha)}(u(1-u))^{\alpha-1}\frac{\Gamma(2\alpha)}{\beta^{2\alpha}} \\ &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)}u^{\alpha-1}(1-u)^{\alpha-1} \end{aligned}$$

for $0 < u < 1$. Notice that U has a $Beta(\alpha, \alpha)$ distribution.

Definition 9.1 (Convergence in Distribution): Suppose X is a random variable with cumulative density function $F(x)$ and the sequence $X_1, X_2, X_3, ...$ of random variables with cumulative density functions $F_1(x)$, $F_2(x)$, $F_3(x)$, ..., respectively. The sequence X_n converges in distribution to X, denoted $X_n \xrightarrow{D} X$ if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for all values x at which $F(x)$ is continuous. The distribution of X is called the limiting distribution of X_n.

Example 9.1: Let $X_2, X_3, X_4, ...$ be a sequence of random variable such that

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that X_n converges in distribution to $Expo(1)$.

Solution:

$$\begin{aligned}\lim_{n\to\infty} F_{X_n}(x) &= \lim_{n\to\infty} 1-\left(1-\frac{1}{n}\right)^{nx}\\ &= 1-\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{nx}\\ &= 1-e^{-x}\end{aligned}$$

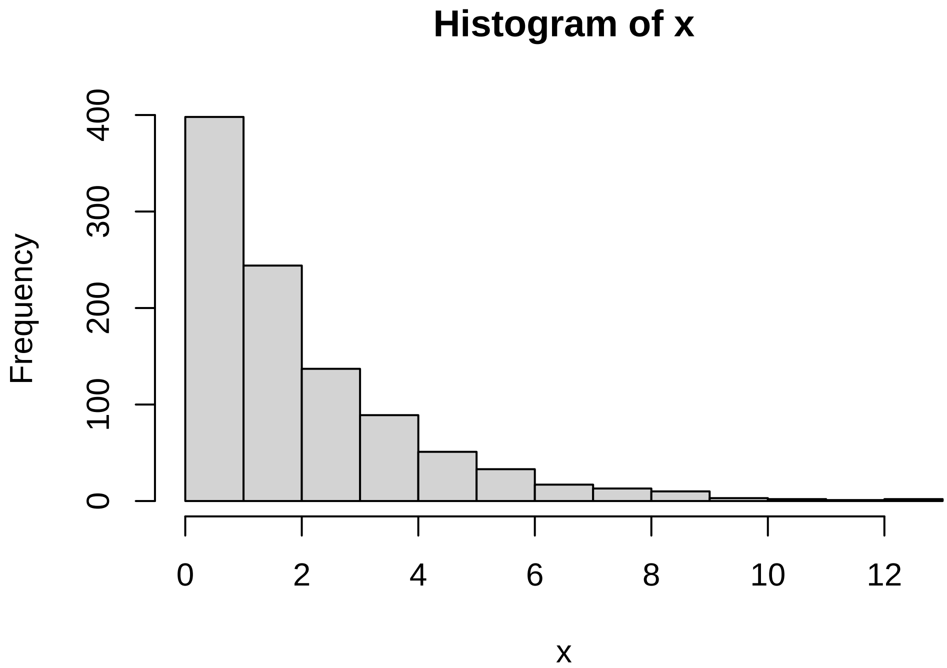
the later is the CDF of exponential distribution with rate $\lambda = 1$. Hence $X_n \xrightarrow{D} X$, and $X \sim Expo(1)$.

Theorem 9.2 (Central Limit Theorem CLT): Let $X_1, X_2, ..., X_n$ be a random sample of size n from a distribution with mean μ and variance $\sigma^2 < \infty$, then the limiting distribution of $Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is $N(0,1)$. That is, Z_n converges in distribution to Z, where $Z \sim N(0,1)$.

Example 9.2 (simulation example): Let $x_1, x_2, ..., x_n$ be a random sample of size $n = 1000$ from $Expo(\lambda)$, for $\lambda = 0.5$. Use R to show that $\bar{X}_n \xrightarrow{D} Y$, where $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$, and $Y \sim N(2,4)$. Also show that $Z_n \xrightarrow{D} Z$, where $Z_n = \frac{\bar{X}_n - \frac{1}{\lambda}}{\frac{1/\lambda}{\sqrt{n}}}$ $Z \sim N(0,1)$.

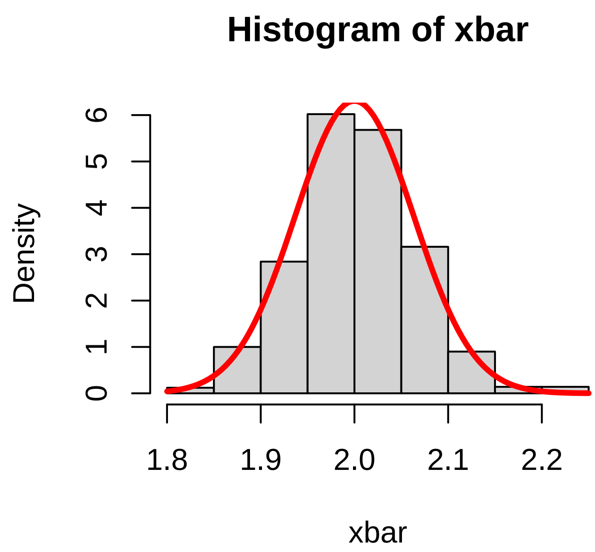
Solution: First, we need to generate a random sample of size 1000 from $Expo(0.5)$ and plot the histogram of the generated sample

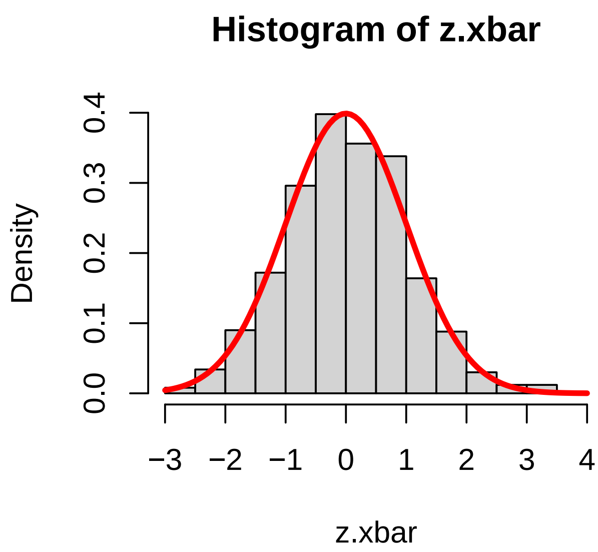
```
set.seed(3)
x <- rexp(1000, rate = 0.5)
hist(x)
```



Then, we need to generate m samples from $Expo(0.5)$, such that m is relatively large (say $m = 1000$) and compute all means $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$. Then, we plot the histogram for $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$ with a $N(2, 4)$ curve overlaid, and plot another histogram for the $(z_1, z_2, ..., z_m)$ with a $N(0, 1)$ curve overlaid.

```
set.seed(3)
xbar <- 0
z.xbar <- 0
for (i in 1:1000){
  x <- rexp(1000, rate = 0.5)
  xbar[i] <- mean(x)
  z.xbar[i] <- (xbar[i] - 2)/(2/sqrt(1000))
}
par(mfrow = c(1, 2))
hist(xbar, freq = F, breaks = 10)
curve(dnorm(x, mean = 2, sd = 2/sqrt(1000)), add = T, col = "red",  lwd = 3)
hist(z.xbar, freq = F, breaks = 10)
curve(dnorm(x), add = T, col = "red",  lwd = 3)
```





Application of the CLT in Hypothesis Tests and Confidence Interval: CLT forms the basis for constructing confidence intervals, performing hypothesis tests, and applying various statistical techniques that rely on the assumption of a normal distribution. Here are some statistical tests that relies on the CLT:

- Suppose $X_1, X_2, ..., X_n$ are a random sample from a Bernoulli distribution with $P(X_i = 1) = 1 - P(X_i = 0) = p$, so that $E[X_i] = p$ and $\text{Var}(X_i) = p(1-p)$. By the CLT,

$$Z = \frac{\bar{X} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

  has an approximate standard normal distribution if n is large. Notice that $\bar{X}$ is considered an estimate of p. Therefore, we can calculate the above statistic under the null hypothesis, where $p = p_0$, about the population proportion p. The CLT allows us to evaluate the test statistic using the standard normal distribution. In addition, we may construct $(1-\alpha)\%$ confidence intervals about

p. That is,

$$\bar{x} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- In linear regression, the quantity

$$t = \frac{\hat{\beta}_1 - \beta_1}{S(\hat{\beta}_1)}$$

follows a t-distribution. We may assume that $\beta_1 = \beta_{1(0)} = 0$ under the null hypothesis about. Therefore, for a sufficiently large n, the t statistic defined above converges in distribution to the standard normal distribution.