Prepared by: Dr. Zaid T. Al-Khaledi

Order Statistics: Let $x_1, x_2, ..., x_n$ be observations from a random sample of size n from a distribution $f(x)$. Let $X_{(1)}$ denote the smallest value of $\{X_1, X_2, ..., X_n\}$, $X_{(2)}$ denote the second smallest value of $\{X_1, X_2, ..., X_n\}$, and similarly $X_{(r)}$ denote the r-th smallest value of $\{X_1, X_2, ..., X_n\}$. Then the random variables $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are called the order statistics of the sample $x_1, x_2, ..., x_n$. Notice that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$, where $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$, and $X_{(n)} = \max\{X_1, X_2, ..., X_n\}$.

The distribution of the order statistics are very important when one uses these in any statistical investigation. The next theorem gives the distribution of an order statistic.

Theorem 10.1: Let $x_1, x_2, ..., x_n$ be a random sample of size n from a distribution with density function $f(x)$. Then the probability density function of the r-th order statistic, $X_{(r)}$, is give as

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x)$$

where $F(x)$ is the CDF of X.

Remark 10.1: The probability density function of the smallest and the largest order statistics are given as

$$f_{X_{(1)}}(x) = n [1 - F(x)]^{n-1} f(x)$$
$$f_{X_{(n)}}(x) = n [F(x)]^{n-1} f(x)$$

Example 10.1: Let x_1, x_2 be a random sample if size 2 from a distribution with the following PDF

$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the density function of $Y = \min\{X_1, X_2\}$?

Solution: The CDF of X is

$$F(x) = \int_0^x e^{-u} du = 1 - e^{-x}$$

notice that $Y = \min\{X_1, X_2\} = X_{(1)}$, and here we have $n = 2$. Hence

$$\begin{aligned} f_Y(y) &= n \left[1 - F(y)\right]^{n-1} f(y) \\ &= 2 \left[1 - \left(1 - e^{-y}\right)\right]^{2-1} e^{-y} \\ &= 2e^{-2y} \end{aligned}$$

for $y \geq 0$.

Example 10.2: Let $Y_1 \leq Y_2 \leq \cdots \leq Y_6$ be the order statistics of a random sample of size 6 from a distribution with the following PDF

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the expected value of Y_6?

Solution: The CDF of X is

$$F(x) = \int_0^x 2u du = x^2$$

notice that $Y_6 = \max\{X_1, X_2, ..., X_6\} = X_{(6)}$, and here we have $n = 6$. Hence

$$\begin{aligned} f_{Y_6}(y) &= n\,[F(y)]^{n-1}\, f(y) \\ &= 6\left[y^2\right]^{6-1}(2y) \\ &= 12y^{11} \end{aligned}$$

for $0 \le y \le 1$.

Therefor the expected value of Y_6 is

$$\begin{aligned} E[Y_6] &= \int_0^1 y f_{Y_6}(y) dy \\ &= \int_0^1 y\left(12y^{11}\right) dy \\ &= \int_0^1 12y^{12} dy \\ &= \frac{12}{13} y^{13}\Big|_0^1 = \frac{12}{13} \end{aligned}$$

Example 10.3: Let x_1, x_2, x_3 be a random sample of size 3 from uniform distribution defined on the interval $(0, a)$. Let $W = \min\{X_1, X_2, X_3\}$. What is the expected value of $\left(1 - \frac{W}{a}\right)^2$?

Solution: The PDF and the CDF of X are

$$f(x) = \begin{cases} \frac{1}{a} & \text{for } 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{a} & \text{for } 0 \le x \le a \\ 1 & \text{for } x \ge a \end{cases}$$

notice that $W = \min\{X_1, X_2, X_3\} = X_{(1)}$, and here we have $n = 3$. Hence

$$\begin{aligned} f_W(w) &= n\left[1 - F(w)\right]^{n-1} f(w) \\ &= 3\left[1 - \frac{w}{a}\right]^{3-1}\left(\frac{1}{a}\right) \\ &= \frac{3}{a}\left(1 - \frac{w}{a}\right)^2 \end{aligned}$$

for $0 \leq w \leq a$.

Therefor the expected value of $\left(1 - \frac{W}{a}\right)^2$ is

$$\begin{aligned} E\left[\left(1 - \frac{W}{a}\right)^2\right] &= \int_0^a \left(1 - \frac{w}{a}\right)^2 f_W(w) dw \\ &= \int_0^a \left(1 - \frac{w}{a}\right)^2 \left(\frac{3}{a}\left(1 - \frac{w}{a}\right)^2\right) dw \\ &= \frac{3}{a}\int_0^a \left(1 - \frac{w}{a}\right)^4 dw \\ &= -\frac{3}{5}\left(1 - \frac{w}{a}\right)^5\Bigg|_0^a \\ &= -\frac{3}{5}\left(\left(1 - \frac{a}{a}\right)^5 - \left(1 - \frac{0}{a}\right)^5\right) = \frac{3}{5} \end{aligned}$$