Stochastic Processes (1)

Lecture 1: Introduction

Vocabulary for Subjects of Stochastic Processes (1)

1) Chapter One: Introduction

- Basic Review to Probability.
- Probability generating function of random variables.
- Probability generating function of sum of fixed number of random variables.
- Probability generating function of sum of random number of random variables.
- Probability generating function of bivariate distribution.

2) Chapter Two: Stochastic Processes

- Introduction to Stochastic Process and definition.
- Specification of Stochastic Processes.
- Classification of Stochastic Processes.
- Introduction to Markov Chain and definition.
- The Initial Distribution and Transition Matrix.
- Higher Order Transition Probability (Chapman Kolmogorov equation).
- Determination of Higher Transition Probability.
- Application and Example

Chapter One: Introduction

(1-1) Definitions:

1) Random Variable:

Consider of an experiment with a probability measure $P(\cdot)$ define on a sample space S and a function that assigns real number to each out come in the sample space of the experiment.

2) Probability Mass Function (p.m.f):

A probability mass function (p.m.f) is a function that gives the probability that a discrete random variable is exactly equal to some value. The probability mass function is often the primary means of defining a discrete probability distribution.

Let X be a discrete random variable with range $S = \{x_1, x_{2,} \dots\}$, (is countable set), then the function:

$P_X(x) = P_r\{X = x\}$, is called the probability mass function (p.m.f) of X.

Then:

1. $\sum P_X(x) = 1, \quad 0 \leq P_X(x) \leq 1$ for all x .
2. For any set $B \subset S$, $P[X \in B] = \sum_{x \in B} P_X(x)$.
3. $F_X(x) = P_r\{X \leq x\}$.
4. $E(x) = \mu_x = \sum_{X \in S} X \, P_X(x)$.

3) Generating Function:

Let $\{a_0, a_1, a_2, \ldots\}$ be a sequence of real numbers, using variables, we define a function:

$$A(S) = a_0 + a_1 S + a_2 S^2 + \cdots = \sum_{k=0}^{\infty} a_k S^k \qquad \ldots 1.1$$

Then $A(S)$ is called generating function of a sequence $\{a_k\}_{k=0}^{\infty}$, $k = 0,1,2, \ldots$, by differentiating k times, putting $(S = 0)$ and dividing by $(k!)$, we get:

$$a_k = \frac{1}{k!} \left[\frac{\partial^k A(S)}{\partial S^k} \right]_{S=0} \qquad \ldots 1.2$$

(1-2) Probability Generating Function:

Suppose that X is a random variable which assumes non-negative integers 0,1,2, ..., and that:

$$P_r\{X = k\} = p_k \quad , \quad k = 0,1,2, \ldots, \qquad \ldots 1.3$$

where : $\sum_{k=0}^{\infty} p_k = 1$

If we take a_k to be the probability p_k , $k = 0,1,2, \ldots$, then the corresponding function $P(S)$ of the sequence of probabilities

$\{p_k\}$ is known as the probability generating function (p.g.f) of the random variable X.

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$
$$= \sum_{k=0}^{\infty} P_r\{X = k\} S^k = E(S^k) \qquad \ldots 1.4$$

Where $E(S^k)$ is the expectation of the function S^k of the random variable X.

The series $P(S)$ converges for at least $-1 \leq S \leq 1$, clearly:

1. $P(1) = 1$
2. The first two derivatives of $P(S)$ are:

$$P'(S) = \sum_{k=1}^{\infty} k\, p_k S^{k-1}, \; -1 \leq S \leq 1$$
$$P''(S) = \sum_{k=2}^{\infty} k(k-1)\, p_k S^{k-2}$$

(1-3) The Mean and Variance of X:

By putting ($S = 1$) in the first derivatives of $P(S)$, we get the expectation of X, i.e:

$$E(X) = P'(1) = \sum_{k=1}^{\infty} k\, p_k \qquad \ldots 1.5$$

Where $E(X)$ is the mean of X. And:

$$P''(1) = \sum_{k=2}^{\infty} k(k-1)\, p_k$$
$$= E[X(X-1)] = E(X^2) - E(X)$$

Then:

$E(X^2) = P''(1) + E(X) = P''(1) + P'(1)$

Hence the variance of X is:

$$var(X) = E(X^2) - [E(X)]^2$$

$$= P''(1) + P'(1) - [P'(1)]^2 \qquad \dots 1.6$$

More generally, the k^{th} factorial moment of X is:

$$E[X(X-1)(X-2)\dots(X-k+1)] = \frac{\partial^k P(S)}{\partial S^k}\bigg]_{S=1}, k = 1,2,\dots$$

Note that: $P(e^S)$ is the moment generating function.