Stochastic Processes (1):

Lecture 10: Higher Transition Probability

(10-1) Higher Transition Probability:

Let p_{ij} denoted the transition probability from state i to state j in one-step transition:

$$p_{ij} = P_r\{x_{n+1} = j | x_n = i\}$$

And the m- Step transition probability is given by:

$$p_{ij}^{(m)} = P_r\{x_{n+m} = j | x_n = i\},$$

is the transition probability from state i to state j in m-step transition, where:

$$p_{ij}^{(2)} = P_r\{x_{n+2} = j | x_n = i\}$$

is a two-step transition probability, means that the state j can be reached from state i in two steps through some intermediate state r.

(10-2) Chapman-Kolmogorov Equation:

Consider a fixed value of r, where:

$$P_r\{x_{n+2} = j, x_{n+1} = r | x_n = i\}$$

$$= P_r\{x_{n+2} = j | x_{n+1} = r, x_n = i\} P_r\{x_{n+1} = r | x_n = i\} \ ,$$

(by Markov property)

$$= P_r\{x_{n+2} = j | x_{n+1} = r\} P_r\{x_{n+1} = r | x_n = i\}$$

$$= p_{rj}^{(1)} . p_{ir}^{(1)} = p_{ir} . p_{rj}$$

Since these intermediate states $r = 1,2,3, \dots$ are mutually exclusive states, we have:

$$p_{ij}^{(2)} = P_r\{x_{n+2} = j | x_n = i\} = \sum_r P_r\{x_{n+2} = j, x_{n+1} = r | x_n = i\}$$

$$= \sum_r p_{ir} . p_{rj} \ ,$$

(Summing over all the intermediate states)

By mathematical induction, we have:

$$p_{ij}^{(m+1)} = P_r\{x_{n+m=1} = j | x_n = i\}$$

$$= \sum_r P_r\{x_{n+m+1} = j, x_{n+m} = r | x_n = i\}$$

$$= \sum_r p_{ir}^{(1)} . p_{rj}^{(m)} = \sum_r p_{ir}^{(m)} . p_{rj}^{(1)}$$

In general:

$$p_{ij}^{(m+n)} = \sum_r p_{ir}^{(n)} . p_{rj}^{(m)} = \sum_r p_{ir}^{(m)} . p_{rj}^{(n)}$$

This equation is special case of Chapman-Kolmogorov equation which is satisfied by the transition probabilities of Markov Chain.

Since:

$$P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0n} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n0} & p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix},$$

Then: $P^{(m)} = \left[p_{ij}^{(m)}\right]$, m-step.

And: $P^{(m+1)} = P^m . P = P . P^m$

In general: $P^{(m+n)} = P^m . P^n = P^n . P^m$

For example:

$$P^{(2)} = P^2 = P . P$$

$$P^{(3)} = P^3 = P^2 . P = P . P^2$$

$$P^{(5)} = P^5 = P^4 . P = P^3 . P^2 = P . P^4$$

Theorem (2): Let P be the transition matrix of a Markov Chain, then the m-step transition matrix is equal to the m^{th} power of P, i.e.:

$$P^{(m)} = P^m$$

Example (10.1):

Consider a M.C. with transition matrix:

$$P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \; S = \{1,2\},$$

then find:

1) 2-step transition matrix.
2) 4-step transition matrix.
3) 5-step transition matrix.
4) $p_{12}^{(2)}, p_{22}^{(4)}, p_{12}^{(1)}$

Solution:

1) $P^{(2)} = P^2 = P.P = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

2) $P^{(4)} = P^4 = P^2.P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} . \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}$

3) $P^{(5)} = P^5 = P^4 . P = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{11}{16} \\ \frac{11}{32} & \frac{21}{32} \end{bmatrix}$

4) $p_{12}^{(2)} = \frac{1}{2}$, $p_{22}^{(4)} = \frac{11}{16}$, $p_{12}^{(1)} = 1$

Example (10.2):

Three students A, B and C are using only one calculator. The student A gives the calculator to the student B or C after he finishing his work, but B gives the calculator to the student C always, so student C flips two coins, if two heads occur, then he gives the calculator to A, otherwise he gives the calculator to B.

1) Write the transition matrix for this example.
2) When the calculator with the student C, what the probability that he will gives to student B.
3) If the calculator with A, what the probability it will be with B after two-step.

Solution: the state space is $S = \{A, B, C\}$

$$1)\ P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} \end{array}$$

$$2)\ P_r\{x_{n+1} = B | x_n = C\} = p_{CB} = \frac{3}{4}$$

$$3)\ p_{AB}^{(2)} = P_r\{x_{n+2} = B | x_n = A\} = P_r\{x_2 = B | x_0 = A\}$$

$$P^{(2)} = P^2 = P.P =$$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} . \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} \end{bmatrix}$$

$$p_{AB}^{(2)} = \frac{3}{8}$$