Stochastic Processes (1):

Lecture 11: Initial Distribution and Probability Distribution

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(11-1) Initial Distribution and Probability Distribution:

Let the stochastic vector: $p^{(0)} = \left[p_0^{(0)} p_1^{(0)} p_2^{(0)} \dots \right]$, with the state space $S = \{0,1,2,\dots\}$, denote the initial probability distribution or the distribution when the process is begins, where:

$$P_r\{x_0 = i\} = p_i^{(0)} \ ; \quad i = 0,1,2,\dots$$

Then, after the first n steps we have:

$p^{(n)} = \left[p_0^{(n)} p_1^{(n)} p_2^{(n)} \dots \right]$ denote the n^{th} step probability distribution with:

$P_r\{x_n = i\} = p_i^{(n)} \ ; \quad i = 0,1,2,\dots$, where:

$$p^{(n)} = p^{(n-1)}.P$$

$$= \left[p_0^{(n-1)} p_1^{(n-1)} p_2^{(n-1)} \dots \right] \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Theorem (3):

Let P be the transition matrix of a M.C., and let $p^{(0)}$ be the initial probability distribution of the process, then:

$p^{(n)} = p^{(0)}P^n$, is the probability distribution of the process after n-steps.

Proof:

$p^{(n)} = p^{(n-1)}P$, then:

$p^{(1)} = p^{(0)}P$, (prob. dist. after one-step)

$p^{(2)} = p^{(1)}P = p^{(0)}P.P = p^{(0)}P^2$, (prob. dist. after two-steps)

$p^{(3)} = p^{(2)}P = p^{(0)}P^2.P = p^{(0)}P^3$, (prob. dist. after three-steps)

$p^{(n)} = p^{(n-1)}.P = p^{(0)}P^{n-1}.P = p^{(0)}P^n$, (prob. dist. after n-steps)

Example (11.1):

If we have a M.C. with state space $S = \{A, B, C\}$ and the initial distribution of the system is: $p^{(0)} = [0 \quad 0 \quad 1]$, with the transition matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$, find: $p^{(1)}, p_A^{(2)}, p_C^{(2)}, P_r\{x_3 = B\}$.

Solution:

1) $p^{(1)} = p^{(0)}P = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

2) $p^{(2)} = p^{(1)}P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$= \left[p_A^{(2)} p_B^{(2)} p_C^{(2)}\right]$, then: $\quad p_A^{(2)} = P_r\{x_2 = A\} = 0$

3) $p_C^{(2)} = \frac{1}{2}$

4) $p^{(3)} = p^{(2)}P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$= \left[p_A^{(3)} p_B^{(3)} p_C^{(3)}\right]$, then: $P_r\{x_3 = B\} = p_B^{(3)} = \frac{1}{4}$

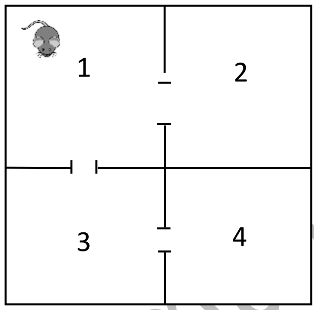
Example (11.2):

Mouse found in one of four rooms shown in the figure below. It moving between rooms a day at a random and not remain in one place. Suppose that x_n is the room operated by mouse. Find:

1) The transition matrix and the state space.
2) If mouse is seen in the second room one day, what is the probability that will be moves to the first room next step.
3) $P_r\{x_2 = 1 |\, x_0 = 4\}$

4) If the initial location of mouse is in the room (1), find the probability distribution after two-steps (two days).

Solution:



1) The state space: $S = \{1,2,3,4\}$

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2) $P_r\{x_1 = 1 | \, x_0 = 2\} = p_{21} = 1$

3) $P_r\{x_2 = 1 | \, x_0 = 4\} = p_{41}^{(2)}$

$$P^{(2)} = P^2 = P.P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$, then: $p_{41}^{(2)} = \frac{1}{2}$

4) Since the mouse is seen in the first room, then the initial distribution will be in the following: $p^{(0)} = [1 \quad 0 \quad 0 \quad 0]$

Then the probability distribution after two-steps (two days) is:

$$p^{(2)} = p^{(1)}P = p^{(0)}P^2 = [1 \quad 0 \quad 0 \quad 0]\begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Example (11.3): H.W

If we have a M.C. with state space $S = \{1,2,3\}$, and the following transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix},$$ with the initial distribution:

$$p^{(0)} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Find: $p^{(2)}, p_{13}^{(2)}, p_3^{(2)}, P_r\{x_3 = 2\}, p_1^{(4)}$.

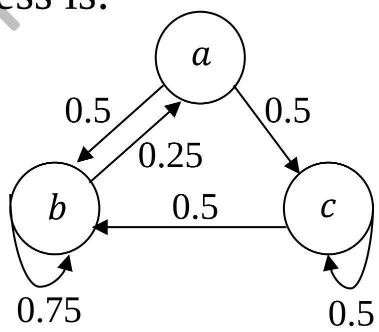
(11-2) Transition Diagram and Transition Tree:

1) Transition Diagram:

The transition probability of Markov Chain can be represented by a diagram call (Transition Diagram), where a probability p_{ij} is denoted by an arrow from state i to state j, for example consider the following transition matrix:

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.75 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, S = \{a, b, c\}$$

Then the transition diagram for this process is:

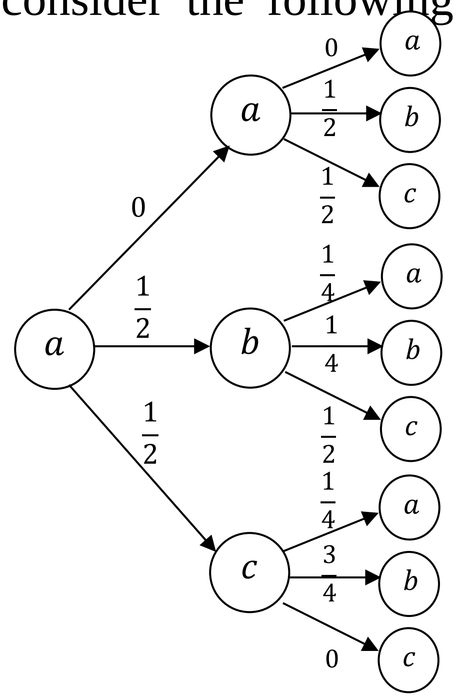


2) Transition Tree:

We can use the transition tree to find the n-step transition from state i to state j, for example consider the following transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}, S = \{a, b, c\}$$

Then find using transition tree:



1- $P_r\{x_2 = b|\, x_0 = a\}$:

$$P_r\{x_2 = b|\, x_0 = a\} = p_{ab}^{(2)} = \left(0 \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right)$$

$$= 0 + \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

2- $P_r\{x_2 = c|\, x_0 = a\} = p_{ac}^{(2)} = \left(0 \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times 0\right)$

$$= 0 + \frac{1}{4} + 0 = \frac{1}{4}$$

3- $P_r\{x_2 = c|\, x_0 = b\} = p_{bc}^{(2)}$

$$= \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times 0\right)$$

$$= \frac{1}{8} + \frac{1}{8} + 0 = \frac{1}{4}$$