Stochastic Processes (1)

Lecture 2: Examples

Example (1.1):

Let X be a Poisson variate with probability mass function:

$$p_k = P_r\{X = k\} = \frac{e^{-\lambda}\ \lambda^k}{k!}\ ,\ k = 0,1,2,\ldots$$

The p.g.f. of Poisson distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\ \lambda^k}{k!} S^k$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda S)^k}{k!}$$

$$= e^{-\lambda}\ e^{\lambda S}\ ,\quad \text{where: } e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

$$= e^{\lambda(S-1)}\ , \text{the p.g.f. of Poisson distribution.}$$

Then the mean and variance of X is:

$E(X) = P'(1)$, then

$P'(S) = \lambda\, e^{\lambda(S-1)}$

Put $S = 1$, we have:

$P'(1) = \lambda\, e^{\lambda(S-1)}\big|_{S=1} = \lambda e^0 = \lambda$, the mean of X.

$var(X) = P''(1) + P'(1) - [P'(1)]^2$

$P''(1) = \lambda^2\, e^{\lambda(S-1)}\big|_{S=1} = \lambda^2$

Then:

$var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$, the variance of X.

Example (1.2):

Let X be a random variable with Geometric distribution p_k where:

$p_k = P_r\{X = k\} = q^k p$, $k = 0,1,2, \dots$.

Find the p.g.f of X and the mean and variance of it.

Solution:

The p.g.f. of Geometric distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=0}^{\infty} q^k p\, S^k$$

$$= p \sum_{k=0}^{\infty} (qS)^k \quad , \quad \text{since} \quad \sum_{k=0}^{\infty} a^K = \frac{1}{1-a}, \text{ then:}$$

$P(S) = \frac{p}{1-qs}$, the p.g.f. of Geometric distribution.

Then the mean and variance of X is:

$$E(X) = P'(1)$$

$$= -p(1-qS)^{-2}(-q)|_{S=1}$$

$$= \frac{pq}{(1-qS)^2}\bigg|_{S=1} = \frac{pq}{p^2} = \frac{q}{p}$$, the mean of X.

$$var(X) = P''(1) + P'(1) - [P'(1)]^2$$

$$P''(1) = -2pq(1 - qS)^{-3}(-q)|_{S=1}$$

$$= \frac{2pq^2}{(1-qS)^3}\bigg|_{S=1} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

Then:

$$var(X) = \frac{2q^2}{p^2} + \frac{q}{p} - (\frac{q}{p})^2$$

$$= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q}{p}\left(1 + \frac{q}{p}\right) = \frac{q}{p}\left(\frac{p+q}{p}\right) = \frac{q}{p^2},$$

the variance of X.

Example (1.3):

Let X be a r.v. with Binomial distribution with (p.m.f) p_k, where:

$$p_k = P_r\{X = k\} = \binom{n}{k} p^k q^{n-k}\ , \quad k = 0,1,2, \ldots$$

Find the mean and variance of X by using p.g.f.

Solution:

The p.g.f. of Binomial distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=0}^{\infty}\binom{n}{k} p^k q^{n-k}\, S^k$$

$$= \sum_{k=0}^{\infty}\binom{n}{k} (pS)^k q^{n-k}$$

from Binomial theory, we have:

$P(S) = (pS + q)^n$, the p.g.f. of Binomial distribution.

Then the mean and variance of X is:

$E(X) = P'(1)$

$$= np(pS + q)^{n-1}|_{S=1}$$

$$= np(p + q)^{n-1} = np$$, the mean of X.

$var(X) = P''(1) + P'(1) - [P'(1)]^2$

$P''(1) = n(n-1)p^2(pS + q)^{n-2}|_{S=1}$

$$= n(n-1)p^2 (p + q)^{n-2} = n(n-1)p^2$$

Then:

$var(X) = n(n-1)p^2 + np - (np)^2$

$$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$$

the variance of X.