Stochastic Processes (1)

Lecture 3: Examples

Example (1.4):

Let X be a Bernoulli variate p_k where:

$p_k = P_r\{X = k\} = p^k(1-p)^{1-k}$, $k = 0, 1$, find the p.g.f of X, and the mean and variance of it.

Solution:

The p.g.f. of Bernoulli distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=0}^{1} p^k(1-p)^{1-k} S^k$$

$$= \sum_{k=0}^{1} (pS)^k(1-p)^{1-k}$$

$$= (pS)^0(1-p)^{1-0} + (pS)^1(1-p)^{1-1}$$

$= (1-p) + pS$, since: $q = 1 - p$

$= q + pS$, the p.g.f. of Bernoulli distribution.

Then the mean and variance of X is:

$$E(X) = P'(1)$$

$= p|_{S=1} = p$, the mean of X.

$$var(X) = P''(1) + P'(1) - [P'(1)]^2$$

$P''(1) = 0|_{S=1} = 0$

Then:

$var(X) = 0 + p - p^2 = p(1-p) = pq$, the variance of X.

Example (1.5):

Suppose that X be a random variable distributed as Geometric distribution with probability function:

$p_k = P_r\{X = k\} = q^{k-1}p$, $k = 1,2, \dots$, find the p.g.f of X, and the mean and variance of it.

Solution:

The p.g.f. of Geometric distribution is:

$$P(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=1}^{\infty} q^{k-1} p\, S^k$$

$$= pS \sum_{k=1}^{\infty} (qS)^{k-1}$$

Let $u = k - 1$, then:

$P(S) = pS \sum_{u=0}^{\infty} (qS)^u$, since $\sum_{k=0}^{\infty} X^K = \frac{1}{1-X}$,

$P(S) = \frac{pS}{1-qs}$, the p.g.f. of Geometric distribution.

Then the mean and variance of X is:

$$E(X) = P'(1)$$

$$= \left.\frac{(1-qS)p-pS(-q)}{(1-qS)^2}\right|_{S=1}$$

$$= \left.\frac{p-pqS+pqS}{(1-qS)^2}\right|_{S=1} = \left.\frac{p}{(1-qS)^2}\right|_{S=1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \quad ,$$

the mean of X.

$$var(X) = P''(1) + P'(1) - [P'(1)]^2$$

$$P''(1) = -2p(1-qS)^{-3}(-q)|_{S=1}$$

$$= \left.\frac{2pq}{(1-qS)^3}\right|_{S=1} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

Then:

$$var(X) = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2q+p-1}{p^2}$$

$$= \frac{2(1-p)+p-1}{p^2} = \frac{2-2p+p-1}{p^2} = \frac{(1-p)}{p^2} = \frac{q}{p^2} \quad ,$$

the variance of X.

Example (1.6): (H.W.)

Let X has a Zero-truncated Poisson distribution with zero class missing:

$$p_k = P_r\{X = k\} = \frac{(e^a - 1)^{-1} a^k}{k!} \;, k = 1,2,\dots$$

Show that:

1. $P(S) = (e^a - 1)^{-1}(e^{as} - 1)$
2. Verify that: $P(1) = \sum_{k=1}^{\infty} p_k = 1$
3. $E(X) = \frac{ae^a}{(e^a - 1)}$

Example (1.7):

Let X be a r.v. with p.g.f. $P(S)$, find the p.g.f of the r.v. $(Y = mX + n)$, where m and n are integers and $m \neq 0$.

Solution:

Let $P_x(S)$ and $P_y(S)$ be the p.g.f. of X and Y respectively, then:

$$P_y(S) = E(S^Y) = E(S^{mX+n})$$

$$= E(S^n . S^{mX}) = S^n E(S^m)^X$$

$$= S^n . P_x(S^m)$$