Stochastic Processes (1)

Lecture 4: P.G.F. of Sum of Fixed Number of r.v.'s

(1-4) P.G.F. of Sum of Fixed Number of Discrete r.v.'s :

Let X and Y be two independent non-negative integer valued random variables with probability distribution given by:

$P\{X = k\} = a_k \quad , \quad P\{Y = j\} = b_j$.

The sum $Z = X + Y$ is r.v. , then the event $[Z = r]$ can be happen in the following mutually exclusive way with corresponding probability :

$(X = 0 \quad and \quad Y = r)$ with prob. $a_o b_r$

$(X = 1 \quad and \quad Y = r - 1)$ with prob. $a_1 b_{r-1}$

$(X = 2 \quad and \quad Y = r - 2)$ with prob. $a_2 b_{r-2}$

$\vdots \qquad \vdots$

$(X = r \quad and \quad Y = 0)$ with prob. $a_r b_0$

Hence the distribution of Z is given by :

$$C_r = P_r\{Z = r\} = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_r b_0 \quad ..1.8$$

$$C_r = \sum_{i=0}^{r} a_i \, b_{r-i} \quad \ldots 1.9$$

Then the new sequence $\{C_r\}$ with results from a combination of the two sequences $\{a_k\}$ and $\{b_j\}$ is called the convolution of $\{a_k\}$ and $\{b_j\}$, and is denoted by:

$$\{C_r\} = \{a_k\} * \{b_j\}$$

Let $P_X(S)$, $P_Y(S)$ and $P_Z(S)$ be the p.g.f.'s of X, Y and Z respectively, then from (1.9) it follows that:

$$P_Z(S) = E(S^{X+Y}) = E(S^X S^Y)$$
$$= E(S^X).E(S^Y) = P_X(S).P_Y(S) \qquad \dots 1.10$$

This is because the independence of X and Y .

Where:

$$P_Z(S) = \sum_{r=0}^{\infty} C_r S^r$$

$$= \sum_{r=0}^{\infty} \sum_{i=0}^{r} a_i b_{r-i} S^r$$

$$= \sum_{r=0}^{\infty} \sum_{i=0}^{r} a_i S^i \,.\, b_{r-i} S^{r-i}$$

$$\left(S^r = S^{r-i+i} = S^i S^{r-i}\right)$$

$$= \sum_{i=0}^{\infty} a_i S^i \sum_{r=0}^{\infty} b_{r-i} S^{r-i}$$

(because the independence of X and Y)

$$= P_X(S).P_Y(S)$$

Theorem (1):

The p.g.f. of the sum two independent random variables X and Y is the product to p.g.f. of X and Y, i.e.:

If : $Z = X + Y$, then:

$$P_Z(S) = P_X(S).P_Y(S) \qquad \ldots 1.11$$

The result with be also hold in the case of the sum (S_n) of (n) non-negtive independent integer valued r.v.'s $X_1, X_2, \ldots, X_n$, i.e. :

If: $S_n = X_1 + X_2 + \cdots + X_n$, then the p.g.f. of S_n is :

$$P_{S_n}(S) = P_{X_1}(S).P_{X_2}(S) \ldots P_{X_n}(S)$$

$$= \prod_{i=1}^{n} P_{X_i}(S) \qquad \ldots 1.12$$

Theorem (2):

The sum $S_n = X_1 + X_2 + \cdots + X_n$ of fixed number (n) of independently and identically distribution (i.i.d.) r.v.'s X_i has p.g.f. as :

$$P_{S_n}(S) = P_{X_1}(S)\, P_{X_2}(S) \ldots P_{X_n}(S)$$

$$P_{S_n}(S) = [P_X(S)]^n \qquad \ldots 1.13$$

Example (1.9):

Let X_1 and X_2 be two independent Poisson variate with means (parameters) $\lambda_1$and λ_2 respectively. Find the p.g.f. of the sum: $Z = X_1 + X_2$ and the mean and variance of Z .

Solution:

We know that the p.g.f. of X_i is:

$$P_{X_i}(S) = \sum_{k=0}^{\infty} p_k S^k \ , i = 1,2$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda_i} \lambda_i^k}{k!} S^k = e^{-\lambda_i} \sum_{k=0}^{\infty} \frac{(\lambda_i S)^k}{k!} = e^{-\lambda_i} \ e^{\lambda_i S}$$

$$= e^{\lambda_i (S-1)} \quad , \ i = 1,2$$

Then the p.g.f. of Z is:

$$P_Z(S) = \prod_{i=1}^{2} P_{X_i}(S) = P_{X_1}(S) \ P_{X_2}(S) \ , \text{[from theorem (1)]}$$

$$= e^{\lambda_1(S-1)} \ e^{\lambda_2(S-1)}$$

$$= e^{(\lambda_1+\lambda_2)(S-1)}$$

$$= e^{\lambda(S-1)} \qquad , \text{where } \lambda = \lambda_1 + \lambda_2$$

Then the distribution of Z is a Poisson distribution with parameter ($\lambda_1 + \lambda_2$),

$Z \sim poi(\lambda_1 + \lambda_2)$.

The mean of Z is:

$$E(Z) = P'_Z(1) = (\lambda_1 + \lambda_2)\, e^{(\lambda_1+\lambda_2)(S-1)}\Big|_{S=1}$$

$$= \lambda_1 + \lambda_2 \quad \text{, the mean of } Z.$$

$$var(Z) = P''_Z(1) + P'_Z(1) - [P'_Z(1)]^2$$

$$P''_Z(1) = (\lambda_1 + \lambda_2)^2\, e^{(\lambda_1+\lambda_2)(S-1)}\Big|_{S=1} = (\lambda_1 + \lambda_2)^2$$

Then:

$$var(Z) = (\lambda_1 + \lambda_2)^2 + (\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2)^2$$

$$= (\lambda_1 + \lambda_2) \quad \text{, the variance of } Z \text{ .}$$