Stochastic Processes (1)

Lecture 5: Examples

Example (1.10):

If $X_1 , X_2 , \ldots , X_n$ be n-independent Poisson distribution with parameters $\lambda_i , i = 1,2,..,n$, then the sum $S_n = \sum_{i=1}^{n} X_i$ is a Poisson distribution with parameter $\sum_{i=1}^{n} \lambda_i$, and $E(S_n) = var(S_n) = \sum_{i=1}^{n} \lambda_i$.

Solution:

Since $X_i \sim poi(\lambda_i)$, then the p.g.f. of X_i is :

$$P_{X_i}(S) = e^{\lambda_i(S-1)} , i = 1,2, \ldots , n$$

Then the p.g.f. of $S_n = \sum_{i=1}^{n} X_i$ is :

$$P_{S_n}(S) = \prod_{i=1}^{n} P_{X_i}(S) = P_{X_1}(S) \; P_{X_2}(S) \ldots P_{X_n}(S)$$

, [from theorem (1)]

$$= e^{\lambda_1(S-1)} \; e^{\lambda_2(S-1)} \ldots e^{\lambda_n(S-1)}$$

$$= e^{\sum_{i=1}^{n} \lambda i(S-1)}$$

$$= e^{\lambda(S-1)} \quad , \quad \lambda = \sum_{i=1}^{n} \lambda_i \text{ , the p.g.f. of } S_n .$$

Then the distribution of S_n is a Poisson dist. with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$.

$$E(S_n) = P'_{S_n}(1) = \sum_{i=1}^{n} \lambda_i \, e^{\sum \lambda i(S-1)}\Big|_{S=1}$$

$$= \sum_{i=1}^{n} \lambda_i \quad \text{, the mean of } S_n \text{ .}$$

$$var(S_n) = P''_{S_n}(1) + P'_{S_n}(1) - [P'_{S_n}(1)]^2$$

$$P''_{S_n}(1) = \left(\sum_{i=1}^{n} \lambda_i\right)^2 e^{\sum \lambda i(S-1)}\Big|_{S=1}$$

$$= \left(\sum_{i=1}^{n} \lambda_i\right)^2$$

Then:

$$var(S_n) = \left(\sum_{i=1}^{n} \lambda_i\right)^2 + \sum_{i=1}^{n} \lambda_i - \left(\sum_{i=1}^{n} \lambda_i\right)^2$$

$$= \sum_{i=1}^{n} \lambda_i \quad \text{, the variance of } S_n \text{ .}$$

Example (1.11):

Let X_1 and X_2 be two independent random variables, each has the same Geometric distribution, with:

$p_k = q^k p$, $k = 0,1,2, \dots , n$, find the p.g.f. of $Z = X_1 + X_2$, and the mean and variance of Z.

Solution:

Since X_i is a geometric distribution with p.g.f.:

$$P_{X_i}(S) = \frac{p}{1-qS} \quad ,i = 1,2$$

Then:

$$P_Z(S) = \left[P_X(S)\right]^n$$, the same parameters [theorem (2)].

$$= \left[P_X(S)\right]^2$$

$$= [\frac{p}{1-qS}]^2$$

$$= \frac{p^2}{(1-qS)^2}$$, the p.g.f. of $(Z = X_1 + X_2)$.

The mean and variance of Z is:

$$E(Z) = P_Z'(1) = -2p^2(1-qS)^{-3}(-q)|_{S=1}$$

$$= 2p^2q\,(1-q)^{-3} = 2p^2qp^{-3}$$

$$= 2\frac{q}{p}$$, the mean of Z .

$$var(Z) = P_Z''(1) + P_Z'(1) - [P_Z'(1)]^2$$

$$P_Z''(1) = -6\,p^2q(1-qS)^{-4}(-q)|_{S=1} = 6p^2q^2(1-q)^{-4}$$

$$= 6p^2q^2p^{-4} = 6\frac{q^2}{p^2}$$

Then:

$$var(Z) = 6\frac{q^2}{p^2} + 2\frac{q}{p} - \left(2\frac{q}{p}\right)^2 = 6\frac{q^2}{p^2} + 2\frac{q}{p} - 4\frac{q^2}{p^2}$$

$$= 2\frac{q^2}{p^2} + 2\frac{q}{p} = 2\frac{q}{p}\left(\frac{q}{p} + 1\right) = 2\frac{q}{p}\left(\frac{q+P}{p}\right)$$

$$= 2\frac{q}{p}\left(\frac{1}{p}\right) = 2\frac{q}{p^2} \quad \text{, the variance of } Z .$$

Example (1.12):

If X_1 and X_2 be two independents r.v.'s with the same Geometric distribution and $\left(p = \frac{2}{3}\right)$. find the p.g.f. of $Y = \sum_{i=1}^{2} X_i$, and find the mean and variance of Y .

Solution:

Since X_i is a geometric distribution with the same p.g.f.:

$P_{X_i}(S) = \frac{p}{1-qS}$, $i = 1,2$, where $p = \frac{2}{3}$, $q = \frac{1}{3}$.

Then the p.g.f of $Y = X_1 + X_2$ is:

$P_Y(S) = \left[P_{X_i}(S)\right]^n$, [from theorem (2)]

$P_Y(S) = \left[P_{X_i}(S)\right]^2 = [\frac{p}{1-qS}]^2 = \frac{p^2}{(1-qS)^2}$, the p.g.f. of Y .

The mean and variance of Y is:

$$E(Y) = P_Y'(S) = -2p^2(1-qS)^{-3}(-q)|_{S=1}$$

$$= \left.\frac{2p^2q}{(1-qS)^3}\right|_{S=1} = \frac{2p^2q}{(1-q)^3} = 2\frac{q}{p} = 2\frac{1/3}{2/3} = 1$$, the mean of Y .

$$var(Z) = P_Y''(1) + P_Y'(1) - [P_Y'(1)]^2$$

$$P_Y''(1) = -6\,p^2q(1-qS)^{-4}(-q)|_{S=1} = 6p^2q^2(1-q)^{-4}$$

$$= 6p^2q^2p^{-4} = 6\frac{q^2}{p^2}$$

Then:

$$var(Y) = 6\frac{q^2}{p^2} + 1 - (1)^2 = 6\frac{q^2}{p^2}$$

$$= 6\frac{(1/3)^2}{(2/3)^2} = 6\left(\frac{1}{4}\right) = \frac{3}{2}$$, the variance of Z .