Stochastic Processes (1):

Lecture 8: Markov process

(8-1) Markov process:

Some stochastic processes are characterized according to dependence relation among the r.v.'s. The Markov Chain (M.C) has a property for the special case of processes for which the state space S and parameter space T are discrete.

Example (8.1):

Consider a simple coin tossing repeating several times, the possible outcomes at each trial are two, head (H) with probability (p), and tail (T) with probability (q) with $(p+q=1)$. Denote head by (1) and tail by (0), and the random variable denoting the result of the n^{th} toss by X_n, then, for $n=0,1,2,\ldots$:

$P_r\{X_n=1\}=p$ and $P_r\{X_n=0\}=q$, thus we have a sequence of r.v.'s $x_1,x_2,\ldots,x_n$, the trials are independent and result of the n^{th} trial dose not independent (dependent) on the previous trial number $(1,2,\ldots,n-1)$.

Now consider the random variable given by the partial sum:

$S_n=x_1+x_2+\ \ldots\ +x_n$, the sum (S_n) gives accumulated number of heads in the first (n) trials and its possible values are $(0,1,2,\ldots,n)$, we have:

$$S_{n+1}=S_n+x_{n+1}$$

Gives that $S_n = j\ , j = 0,1,2, \dots , n$, the r.v. S_{n+1} can assume only two possible values $S_{n+1} = j$ with probability (q) and $S_{n+1} = j + 1$ with probability (p), the probabilities are not all affected by the values of the r.v.'s $S_1, S_2, \dots$, thus :

$$P_r\{S_{n+1} = j + 1 | S_n = j\} = p$$
$$P_r\{S_{n+1} = j | S_n = j\} = q$$

This is an example of Markov Chain, a case of simple dependence that outcome of $(n+1)^{th}$ trail depends directly on that of the $(n)^{th}$ trial and only on it.

(8-2) Definition of Markov Chain:

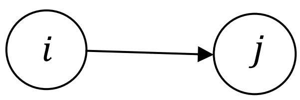
The stochastic processes $\{X_n, n = 0,1,2, \dots\}$ with discrete state space S and discrete parameter space T is called a Markov Chain if for $i, j, i_0, i_1, \dots , i_{n-1} \in N$, the set of all integers:

$$P_r\{x_{n+1} = j | x_n = i, x_{n-1} = i_{n-1}, \dots , x_0 = i_0\}$$
$$= P_r\{x_{n+1} = j | x_n = i\} = p_{ij}$$

Where p_{ij} is the conditional probability, and it's the probability of transition from state i to state j at n^{th} trail. Then the probabilities p_{ij} are the basic to study the structure of the Markov Chain.

The Transition probability p_{ij} is called One-Step Transition probability if:

$$p_{ij} = P_r\{x_{n+1} = j | x_n = i\}$$



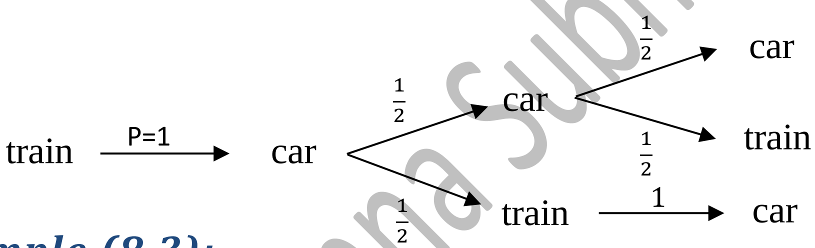
And the m- Step Transition probability is:

$$p_{ij}^{(m)} = P_r\{x_{n+m} = j | x_n = i\} .$$

Example (8.2):

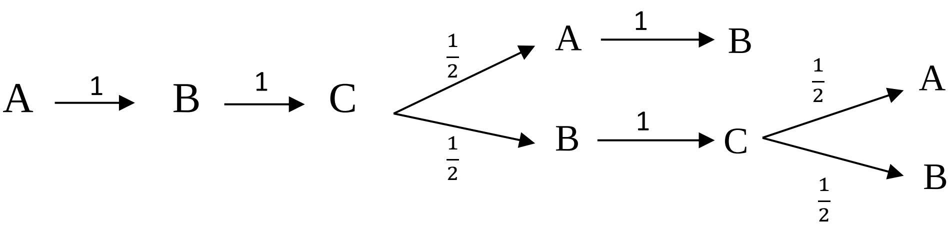
A man either drives his car or catches a train to work each day. Suppose that he never goes by train two days in a row, but if he drives to work, then the next day he is just as likely to drive again as he is to travel by train.

The state space of this system is $\{t(train), d(drive)\}$, this stochastic process is a Markov Chain since the outcomes on any day depends only on what happened the preceding day.



Example (8.3):

Three boys A, B and C are throwing a ball to each other. A always throws the ball to B, and B always throws the ball to C, but C is just likely to throw the ball to B as to A. Let X_n denote the n^{th} person to be thrown the ball. The state space of the system is {A, B, C}. This process is a Markov Chain since the person throwing the ball is not influenced by those who previously had the ball.



(8-3) Transition Probability Matrix:

The transition probabilities $\{p_{ij}\}$ satisfy the conditions that:

$p_{ij} = P_r\{x_{n+1} = j | x_n = i\}$

$p_{ij} \geq 0 \ , \ \sum_{j=o}^{\infty} p_{ij} = 1 \quad \forall\, i$.

These probabilities may be written in the matrix form P as:

$$P = \overset{\quad n+1}{n\begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0n} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n0} & p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}} \ ,$$

$S = \{0,1,2,3, \dots\}, T = \{0,1,2,3, \dots\}$

This matrix is called the transition probability matrix of Markov Chain or (One-Step transition matrix), which is a square matrix with non-negative element and unit row sums.

Example (8.4):

From example (2.7), the transition matrix is:

$$P = \overset{n+1}{n \begin{matrix} \\ A \\ B \\ C \end{matrix}\begin{matrix} A & B & C \\ \left[\begin{matrix} 0 \\ 0 \\ 0.5 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0.5 \end{matrix} & \left.\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right] \end{matrix}} \ , \ S = \{A, B, C\}, T = \{0,1,2,3, \dots\}$$

Theorem (1): Let p_{ij} denoted the transition probability from state i to state j in one-step transition, and let the initial distribution of x_0 is:

$P_r\{x_0 = a\} = p_a$, then:

$$P_r\{x_n = j, x_{n-1} = k, x_{n-2} = i, \dots, x_1 = b, x_0 = a\} = p_{kj}.p_{ik} \ \dots \ p_{ab}.p_a$$

Proof: $P_r\{x_n = j, x_{n-1} = k, x_{n-2} = i, \dots, x_1 = b, x_0 = a\}$

$$= P_r\{x_n = j | x_{n-1} = k, x_{n-2} = i, \dots, x_1 = b, x_0 a\}.P_r\{x_{n-1} = k, x_{n-2} = i, \dots, x_1 = b, x_0 = a\}$$

$= P_r\{x_n = j|x_{n-1} = k\}.P_r\{x_{n-1} = k, x_{n-2} = i, \dots, x_1 = b, x_0 = a\}$, (from Markov property)

$= P_r\{x_n = j|\, x_{n-1} = k\}.P_r\{x_{n-1} = k \mid x_{n-2} = i, \dots, x_1 = b, x_0 = a\}.P_r\{x_{n-2} = i, \dots, x_1 = b, x_0 = a\}$

$= P_r\{x_n = j|\, x_{n-1} = k\}.P_r\{x_{n-1} = k \mid x_{n-2} = i\}.P_r\{x_{n-2} = i, \dots, x_1 = b, x_0 = a\}$

$= P_r\{x_n = j|\, x_{n-1} = k\}.P_r\{x_{n-1} = k \mid x_{n-2} = i\} \dots P_r\{x_1 = b|x_0 = a\}.P_r\{x_0 = a\}$

$= p_{kj} \cdot p_{ik} \;\cdots\; p_{ab} \cdot p_a$

Example (8.5): Let $\{X_n, n = 0,1,2, \dots\}$ be a Markov Chine with state space $S = \{1,2,3\}$ and the transition matrix is:

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix},$$

and the initial distribution $P_r\{x_0 = i\} = \frac{1}{3}$, $i = 1,2,3$, then find the following:

1. $P_r\{x_1 = 2|\, x_0 = 3\} = p_{32} = \frac{3}{4}$
2. $P_r\{x_{n+1} = 3|\, x_n = 2\} = p_{23} = \frac{1}{4}$
3. $P_r\{x_2 = 2, x_1 = 1, x_0 = 2\}$
   $= P_r\{x_2 = 2|x_1 = 1\}.P_r\{x_1 = 1|x_0 = 2\}.P_r\{x_0 = 2\} = p_{12} \cdot p_{21} \cdot p_2$
   $= \frac{1}{4} \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{48}$
4. $P_r\{x_2 = 1, x_1 = 2|\, x_0 = 2\} = \frac{P_r\{x_2=1|x_1=2\}P_r\{x_1=2|\, x_0=2\}P_r\{\, x_0=2\}}{P_r\{\, x_0=2\}}$
   $= P_r\{x_2 = 1|x_1 = 2\}P_r\{x_1 = 2|\, x_0 = 2\} = p_{21} \cdot p_{22} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$
5. $P_r\{x_3 = 2, x_2 = 3, x_1 = 2, x_0 = 3\}$
   $= P_r\{x_3 = 2|x_2 = 3\}.P_r\{x_2 = 3|x_1 = 2\}.P_r\{x_1 = 2|x_0 = 3\}\, P_r\{x_0 = 3\}$
   $= p_{32} \cdot p_{23} \cdot p_{32} \cdot p_3$
   $= \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{3}{64}$
6. $P_r\{x_2 = 3|x_1 = 3, x_0 = 1\} = P_r\{x_2 = 3|x_1 = 3\} = p_{33} = \frac{1}{4}$