Stochastic Processes (1):

Lecture 9: Random Walk

(9.1) Random Walk:

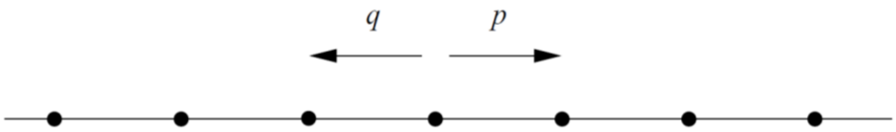
A random walk is a stochastic sequence $\{S_n\}$, with $S_0 = 0$, defined by:

$S_n = \sum_{i=1}^{n} X_i$, where $\{X_i\}$ are independent and identically distributed random variables (i.i.d.).

If $X_i = \mp 1$, with:

$P_r\{x_i = 1\} = p$ and $P_r\{x_i = -1\} = 1 - p = q$, then the process is *Simple random walk.* A simple random walk is symmetric if the particle has the same probability for each of the neighbors. i.e.: $\left(p = \frac{1}{2}, q = \frac{1}{2}\right)$, then it called *Simple symmetric random walk.*

Suppose a particle performing as a simple random walk on the integer points of the real line, where it in each step moves to one of its neighboring points:



Simple random walk

In general, we can also study random walks in higher dimensions. In two dimensions, each point has 4 neighbors and in three dimensions there are 6 neighbors.

Imagine a person or a particle on an axis, so that at each discrete time step, the walker moves either one unit to the right (with probability p) or one unit to the left (with probability $1-p$), independently from step to step. The walker could accomplish this by tossing a coin with probability of heads p at each step, to determine whether to move right or move left.

Example (9.1):

A particle performs a random walk with absorbing barriers at 0 and 4. It's at any position $(0 < i < 4)$, it moves to $(i+1)$ with probability p, or to $(i-1)$ with probability q, $p+q=1$, but as soon as it reaches (0) or (4) it remains there itself. Let X_n be the position of the particle after n-moves. The different states of X_n are the different positions of the particle. This is also a M.C and the transition matrix is given by:

$$P = \begin{array}{c|ccccc} n-1 \backslash n & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 \\ 3 & 0 & 0 & q & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array} \quad , \quad S = \{0,1,2,3,4\}$$

Since:

$$P_r\{x_n = i+1 \mid x_{n-1} = i\} = p$$

$$P_r\{x_n = i-1 \mid x_{n-1} = i\} = q$$

$$P_r\{x_n = 0 \mid x_{n-1} = 0\} = P_r\{x_n = 4 \mid x_{n-1} = 4\} = 1$$

Then find:

1. $P_r\{x_3 = 1 \mid x_2 = 2\} = p_{21} = q$
2. $P_r\{x_3 = 2, x_2 = 1, x_1 = 4, x_0 = 3\}$

   $= P_r\{x_3 = 2 | x_2 = 1\}. P_r\{x_2 = 1 | x_1 = 4\}. P_r\{x_1 = 4 | x_0 = 3\}\ P_r\{x_0 = 3\}$

   $= p_{12}\ .p_{41}\ .p_{34}\ .p_3 = p.0.p.\frac{1}{5} = 0$

3. $P_r\{x_3 = 2|\, x_2 = 3, x_1 = 4, x_0 = 3\} = P_r\{x_3 = 2|\, x_2 = 3\} = p_{32} = q$

4. $P_r\{x_3 = 4, x_2 = 3, x_1 = 2|x_0 = 3\}$

$$= \frac{P_r\{x_3 = 4|x_2 = 3\}P_r\{x_2 = 3|\, x_1 = 2\}P_r\{x_1 = 2|\, x_0 = 3\}P_r\{\, x_0 = 3\}}{P_r\{\, x_0 = 3\}}$$

$$= P_r\{x_3 = 4|x_2 = 3\}P_r\{x_2 = 3|\, x_1 = 2\}P_r\{x_1 = 2|\, x_0 = 3\}$$

$$= p_{34}\,.\,p_{23}\,.\,p_{32} = p.\,p.\,q = p^2q$$