Stochastic Processes (2):

Lecture 1: Initial Distribution and Probability Distribution

(1-1) Initial Distribution and Probability Distribution:

Let the stochastic vector: $p^{(0)} = \left[p_0^{(0)} p_1^{(0)} p_2^{(0)} \dots \right]$, with the state space $S = \{0,1,2,\dots\}$, denote the initial probability distribution or the distribution when the process is begins, where:

$$P_r\{x_0 = i\} = p_i^{(0)} \;\; ; \quad i = 0,1,2,\dots$$

Then, after the first n steps we have:

$p^{(n)} = \left[p_0^{(n)} p_1^{(n)} p_2^{(n)} \dots \right]$ denote the n^{th} step probability distribution with:

$P_r\{x_n = i\} = p_i^{(n)} \;\; ; \quad i = 0,1,2,\dots$, where:

$$p^{(n)} = p^{(n-1)}.P$$

$$= \left[p_0^{(n-1)} p_1^{(n-1)} p_2^{(n-1)} \dots \right] \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Theorem (1):

Let P be the transition matrix of a M.C., and let $p^{(0)}$ be the initial probability distribution of the process, then:

$p^{(n)} = p^{(0)}P^n$, is the probability distribution of the process after n-steps.

Proof:

$p^{(n)} = p^{(n-1)}P$, then:

$p^{(1)} = p^{(0)}P$, (prob. dist. after one-step)

$p^{(2)} = p^{(1)}P = p^{(0)}P.P = p^{(0)}P^2$, (prob. dist. after two-steps)

$p^{(3)} = p^{(2)}P = p^{(0)}P^2.P = p^{(0)}P^3$, (prob. dist. after three-steps)

$p^{(n)} = p^{(n-1)}.P = p^{(0)}P^{n-1}.P = p^{(0)}P^n$, (prob. dist. after n-steps)

Example (1.1):

If we have a M.C. with state space $S = \{A, B, C\}$ and the initial distribution of the system is: $p^{(0)} = [0 \quad 0 \quad 1]$, with the transition matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$, find: $p^{(1)}, p_A^{(2)}, p_C^{(2)}, P_r\{x_3 = B\}$.

Solution:

1) $p^{(1)} = p^{(0)}P = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

2) $p^{(2)} = p^{(1)}P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$= \left[p_A^{(2)} \; p_B^{(2)} \; p_C^{(2)}\right]$, then: $\quad p_A^{(2)} = P_r\{x_2 = A\} = 0$

3) $p_C^{(2)} = \frac{1}{2}$

4) $p^{(3)} = p^{(2)}P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$= \left[p_A^{(3)} p_B^{(3)} p_C^{(3)}\right]$, then: $P_r\{x_3 = B\} = p_B^{(3)} = \frac{1}{4}$

Example (1.2):

If we have the state space $S = \{1,2,3,4\}$, and transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

with initial distribution $p^{(0)} = [1 \quad 0 \quad 0 \quad 0]$.

Find the probability distribution after two-steps.

Solution:

$$P^{(2)} = P^2 = P.P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Then the probability distribution after two-steps is:

$$p^{(2)} = p^{(1)}P = p^{(0)}P^2 = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Example (1.3): H.W

If we have a M.C. with state space $S = \{1,2,3\}$, and the following transition matrix:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{with the initial distribution:}$$

$$p^{(0)} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

Find: $p^{(2)}, p_{13}^{(2)}, p_3^{(2)}, P_r\{x_3 = 2\}, p_1^{(4)}$.