Stochastic Processes (2)

Lecture 3: Classification of State of Markov Chains

(3-1) Definition of the First Passage:

Suppose that a system starts with state j, and let $f_{jk}^{(n)}$ be the probability that it reaches the state k for the first time at n^{th} step (or after n transition), and let $p_{jk}^{(n)}$ be the probability that it reaches the state k (not necessarily for the first time) after n^{th} steps, then the probability of the first passage is:

$$f_{jk}^{(n)} = P_r\{X_n = k, X_r = i; r = 1,2,\dots,n-1 | X_0 = j\ \}$$

and:

$$f_{jj}^{(n)} = P_r\{X_n = j, X_r = i; r = 1,2,\dots,n-1 | X_0 = j\ \}$$

Let F_{jk} denote the probability that the system starting with state j , the system will ever reach the state k as:

$$F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

Then we have two cases: $F_{jk} = 1 \quad or \quad F_{jk} < 1$

If $F_{jk} = 1$, this means that the system is in state j and certainty it will reach to state k.

And if $F_{jk} < 1$, this means that the system is in state j and uncertainty it will reach to state k.

Then the mean of the first passage time from state j to state k is given by:

$$\mu_{jk} = \sum_{n=1}^{\infty} n\, f_{jk}^{(n)}$$

where $\left\{ f_{jk}^{(n)}, n = 1,2, \dots \right\}$ the probability of the first passage.

(3-2) Classification of states of Markov chain:

1) Recurrent and Transient states:

If $j = k$, then F_{jj} will represented the distribution of the recurrent times of state j , then:

- If $F_{jj} = 1$, then the state j is said to be a recurrent (i.e., return to state j is certain).
- If $F_{jj} < 1$, then the state j is said to be a transient (i.e., return to state j is uncertain).

2) Positive and Null recurrent states:

- A recurrent state is said to be Positive recurrent if the mean recurrent μ_{jj} for the state j is finite (i.e., $\mu_{jj} < \infty$).
- A recurrent sate is said to be Null recurrent if the mean recurrent μ_{jj} for the state is infinite (i.e., $\mu_{jj} = \infty$).

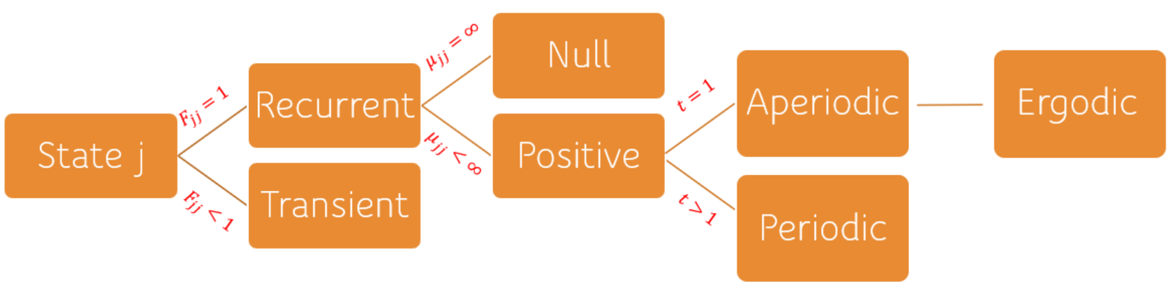
3) Periodic and Aperiodic states:

- A state j is s said to be periodic with period t $(t > 1)$ is return to the state is possible only at $(t, 2t, 3t, \dots)$ steps, when t is greatest integer with this property.
- A state j is said to be aperiodic (or non-periodic) if no such $(t \geq 1)$ exists, for example $(t = 1)$.

4) Ergodic:

A recurrent positive and aperiodic state of the Markov chain is said to be ergodic. A Markov chain with the states ergodic is said to be ergodic chain.

Classification of states of Markov chain



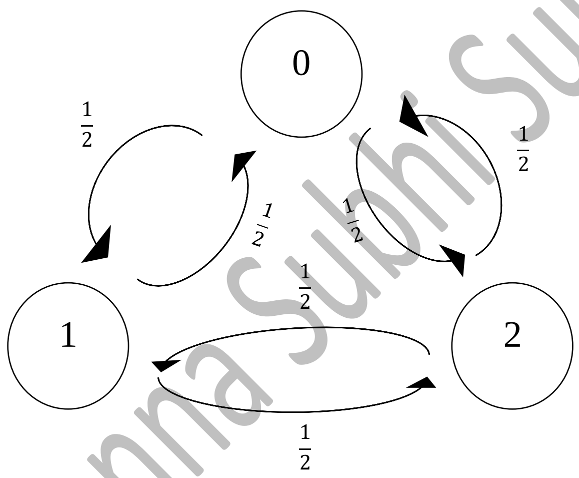
state may be or may not be periodic.

Example (3.1):

If we have a Markov chain with state space {0,1,2} and transition matrix, classify the states of this Markov chain:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution:



State (0):

1. Probability distribution of the first passage for the state is:

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$$f_{00}^{(1)} = 0$$

$$f_{00}^{(2)} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$f_{00}^{(3)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$f_{00}^{(4)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$F_{00} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + f_{00}^{(4)} + \cdots$$

$$F_{00} = 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Since the prob. dist. of the first passage of state (0) is ($F_{00} = 1$), then the state (0) is **recurrent state**.

2. The mean recurrent time of the state (0) is:

$$\mu_{00} = \sum_{n=1}^{\infty} n\, f_{00}^{(n)}$$

$$= 1(0) + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots$$

$$= 0 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots$$

$$= \sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = 2\sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^n + 1 - 1$$

$$= 2\left[\sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^n + \frac{1}{2} - \frac{1}{2}\right]$$

$$= 2\left[\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n - \frac{1}{2}\right]$$

$$= 2\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n - 1 = 2\left(\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}\right) - 1$$

$$= 2\left(\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2}\right) - 1 = 3 < \infty$$

Then the state (0) is **positive recurrent**.

3. $(0) \to (1) \to (0)\ ,\ \ t = 2$

$(0) \to (1) \to (2) \to (0)\ ,\ \ t = 3$

$(0) \to (1) \to (2) \to (1) \to (0)\ ,\ \ t = 4$

Since it returns to state (0) at $(t = 2,3,4,\dots)$ steps, then the state (0) is **aperiodic**.

4. Since the state (0) is positive recurrent with aperiodic state, then the state (0) is **ergodic**.

And also, state (1) and state (2).

Since all states of M.C. are ergodic then the chain is **ergodic chain**.