Stochastic Processes (2)

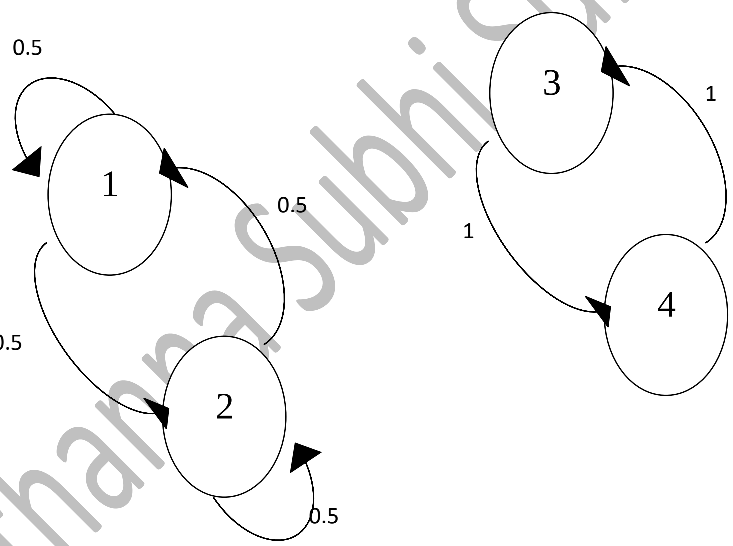
Lecture 4: Examples of classification (Chan & states)

Example (4.1):

Classify the states of this Markov chain.

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ S = \{1,2,3,4\}$$

Solution:



<u>State</u> (1):

1. Probability distribution of the first passage for the state is:

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)}$$

$$f_{11}^{(1)} = 0.5$$

$$f_{11}^{(2)} = 0.5 \times 0.5 = (0.5)^2$$

$$f_{11}^{(3)} = 0.5 \times 0.5 \times 0.5 = (0.5)^3$$

$$f_{11}^{(4)} = 0.5 \times 0.5 \times 0.5 \times 0.5 = (0.5)^4$$

$$\vdots$$

$$F_{11} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + f_{11}^{(4)} + \cdots$$

$$= 0.5 + (0.5)^2 + (0.5)^3 + (0.5)^4 + \cdots = \frac{0.5}{1-0.5} = 1$$

Since the prob. dist. of the first passage of state (1) is ($F_{11} = 1$), then the state (1) is **recurrent state**.

2. The mean recurrent time of the state (1) is:

$$\mu_{11} = \sum_{n=1}^{\infty} n\, f_{11}^{(n)}$$

$$= 1(0.5) + 2(0.5)^2 + 3(0.5)^3 + \cdots$$

$$= \sum_{n=1}^{\infty} n\, (0.5)^n$$

$$= \frac{0.5}{(1-0.5)^2} = 2 < \infty$$

Since the mean recurrent time (μ_{11}) is exist, then the state (1) is **positive recurrent**.

3. Since $t = 1$, return to state (1) in one step, then the state (1) is **aperiodic**.

4. Since the state (1) is positive recurrent with aperiodic state, then the state (1) is **ergodic**.

And also, state (2) is **ergodic**.

State (3):

1. Probability distribution of the first passage for the state is:

$F_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)}$

$f_{33}^{(1)} = 0$

$f_{33}^{(2)} = 1 \times 1 = 1$

$f_{33}^{(3)} = 0$

$f_{33}^{(4)} = 0$

$\vdots$

$F_{33} = f_{33}^{(1)} + f_{33}^{(2)} + f_{33}^{(3)} + f_{33}^{(4)} + \cdots$

$\therefore F_{33} = 0 + 1 + 0 + 0 + \cdots = 1$

Since the prob. dist. of the first passage of state (3) is $(F_{33} = 1)$, then the state (3) is **recurrent state**.

2. The mean recurrent time of the state (3) is:

$\mu_{33} = \sum_{n=1}^{\infty} n\, f_{33}^{(n)}$

$= 1(0) + 2(1) + 3(0) + \cdots = 2 < \infty$

Then the state (3) is **positive recurrent**.

3. $(3) \rightarrow (4) \rightarrow (3), \ \ t = 2$

Since $t > 1$, return to state (3) in two steps, then the state (3) is periodic with **period** $(t = 2)$.

4. Since the state (3) is positive recurrent with periodic state, then the state (3) is **not ergodic**.

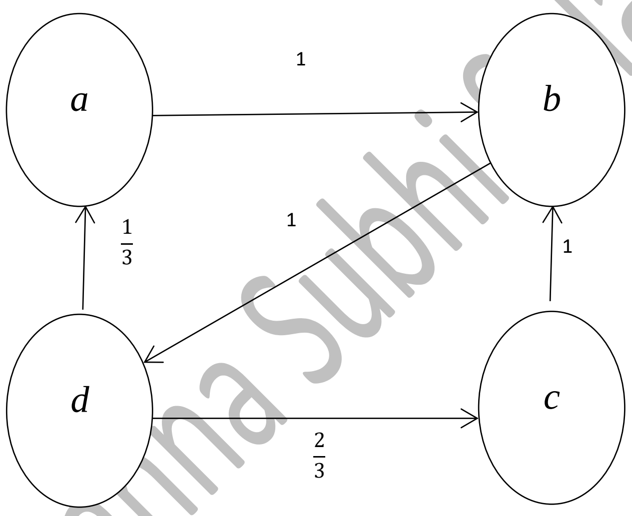
And also, state (4) is **not ergodic**.

Example (4.2):

If we have a Markov Chain with this transition diagram:

a) Classify the chain.

b) Classify all the states of this chain.



Solution:

a) Classify the chain.

1. Closed set are: $C_1 = \{a, b, c, d\}$.
2. Since there is one closed set, then the chain is irreducible.
3. No absorbing state, because no closed set has one state.
4. States $\{a, b, c, d\}$ are accessibility because $p_{ij}^{(n)} > 0$, for some $n \geq 0$.
5. No communicate states in this chain.

b) Classify all the states of this chain.

State (a)

1- Probability distribution of the first passage for the state is:

$$F_{aa} = \sum_{n=1}^{\infty} f_{aa}^{(n)}$$

$$f_{aa}^{(1)} = 0 \ , \ f_{aa}^{(2)} = 0$$

$$f_{aa}^{(3)} = 1 \times 1 \times \frac{1}{3} = \frac{1}{3} \quad , f_{aa}^{(4)} = 0 \ , \ f_{aa}^{(5)} = 0$$

$$f_{aa}^{(6)} = 1 \times 1 \times \frac{2}{3} \times 1 \times 1 \times \frac{1}{3} = \frac{1}{3}\left(\frac{2}{3}\right)$$

$$f_{aa}^{(9)} = 1 \times 1 \times \frac{2}{3} \times 1 \times 1 \times \frac{2}{3} \times 1 \times 1 \times \frac{1}{3} = \frac{1}{3}\left(\frac{2}{3}\right)^2$$

$$f_{aa}^{(12)} = \frac{1}{3}\left(\frac{2}{3}\right)^3 \ , \ \ldots$$

$$\therefore F_{aa} = f_{aa}^{(1)} + f_{aa}^{(2)} + f_{aa}^{(3)} + f_{aa}^{(4)} + \cdots$$

$$F_{aa} = 0 + 0 + \frac{1}{3} + 0 + 0 + \frac{1}{3}\left(\frac{2}{3}\right) + 0 + 0 + \frac{1}{3}\left(\frac{2}{3}\right)^2 + \cdots$$

$$= \frac{1}{3}\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots\right)$$

$$= \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3}\left(\frac{1}{\frac{1}{3}}\right) = 1 \quad ,$$

where $\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots = \frac{1}{1-a}$

Since the prob. dist. of the first passage of state (a) is ($F_{aa} = 1$), then the state (a) is **recurrent state**.

2- The mean recurrent time of the state (a) is:

$$\mu_{aa} = \sum_{n=1}^{\infty} n\, f_{aa}^{(n)}$$

$$= 1(0) + 2(0) + 3\left(\frac{1}{3}\right) + 4(0) + 5(0) + 6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \cdots + 9\frac{1}{3}\left(\frac{2}{3}\right)^2 + \cdots$$

$$= 3\left(\frac{1}{3}\right)\left(1 + 2\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \cdots\right)$$

Where $\sum_{n=1}^{\infty} n\, a^n = a + 2a^2 + 3a^3 + \cdots = \frac{a}{(1-a)^2}$

multiply by $\frac{2}{3}$ and divide by $\frac{2}{3}$, we have:

$$\mu_{aa} = \frac{3}{2}\left(\frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \cdots\right),$$

$$= \frac{3}{2}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{3}{2}\,\frac{\frac{2}{3}}{\left(1-\frac{2}{3}\right)^2} = \frac{1}{\left(\frac{1}{3}\right)^2} = 9 < \infty$$

Then the state (a) is **positive recurrent** because $\mu_{aa} < \infty$.

3- periodic or aperiodic of the state (a):

$(a) \to (b) \to (d) \to (a), \quad t = 3$

$(a) \to (b) \to (d) \to (c) \to (b) \to (d) \to (a), \quad t = 6$

$\vdots$

Since it returns to state (a) at $(t = 3,\ 6,\ 9, \dots)$ steps, then the state (a) is **periodic**.

4- ergodic of state (a):

Since the state (a) is positive recurrent with periodic state, then the state (a) is **not ergodic**.

And also, state (c), (have the same transition).

State (b)

1- Probability distribution of the first passage for the state is:

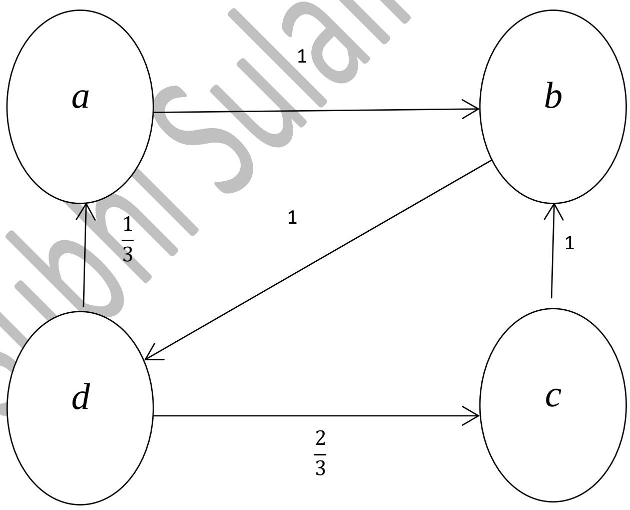
$$F_{bb} = \sum_{n=1}^{\infty} f_{bb}^{(n)}$$

$$f_{bb}^{(1)} = 0 \quad , \quad f_{bb}^{(2)} = 0$$

$$f_{bb}^{(3)} = 1 \times \frac{1}{3} \times 1 + 1 \times \frac{2}{3} \times 1 = 1$$

$$f_{bb}^{(4)} = 0 \quad , f_{bb}^{(5)} = 0$$

$$\vdots$$



$$\therefore F_{bb} = f_{bb}^{(1)} + f_{bb}^{(2)} + f_{bb}^{(3)} + f_{bb}^{(4)} + \cdots$$

$$F_{bb} = 0 + 0 + 1 + 0 + \ \ldots = 1$$

Since the prob. dist. of the first passage of state (b) is ($F_{bb} = 1$), then the state (b) is **recurrent state**.

2- The mean recurrent time of the state (b) is:

$$\mu_{bb} = \sum_{n=1}^{\infty} n \ f_{bb}^{(n)}$$

$$= 1(0) + 2(0) + 3(1) + 4(0) + 5(0) + \cdots$$

$$= 3 < \infty$$

Then the state (b) is **positive recurrent** because $\mu_{bb} < \infty$.

3- periodic or aperiodic of the state (b):

$$(b) \rightarrow (d) \rightarrow (a) \rightarrow (b), \quad t = 3$$

$$(b) \rightarrow (d) \rightarrow (c) \rightarrow (b), \quad t = 3$$

Since it returns to state (b) at $(t = 3)$ steps, then the state (b) is **periodic**.

4- ergodic of state (b):

Since the state (b) is positive recurrent with periodic state, then the state (b) is **not ergodic**.

And also, state (d), (have the same transition).