Stochastic Processes (2)

Lecture 5: Stationary Distribution of Markov Chain

(5-1) Definition:

Consider an irreducible positive recurrent and aperiodic M.C. (i.e., ergodic chain), then the probability distribution $\{\pi_j\}$ is called the stationary distribution of this chain if the system of this linear equation:

$$\pi_j = \sum_{j \in s} \pi_j \, p_{jj} \quad , \quad j \in s$$

where:

$$\sum_{j \in s} \pi_j = 1 \quad and \quad \pi_j \geq 0$$

has a solution: $\underline{\Pi} = [\pi_1 \quad \pi_2 \quad \cdots \quad]$

is exist solution and there is one solution, and:

$$\pi_j = \lim_{n \to \infty} {p_{jj}}^{(n)}$$

Theorem (2):

The stationary distribution $\underline{\Pi}$ is verify the equation:

$$\underline{\Pi}(I - P) = 0$$

where I is an identity matrix.

Proof:

From Chapman-Kolmogorov equation, we have:

$$P^{(n)} = P.P^{(n-1)}$$

Since:

$$\underline{\Pi} = \lim_{n\to\infty} P^{(n)} \text{ , then:}$$

$$\Pi = \lim_{n\to\infty} P^{(n-1)}P$$

$$\underline{\Pi} = \underline{\Pi}\,.P \quad \Rightarrow \quad \underline{\Pi} - \underline{\Pi}\,.P = \underline{0} \text{ , then:}$$

$$\underline{\Pi}(I-P) = \underline{0} \qquad \text{, this is proof of theorem.}$$

Example (5.1):

Let a Markov chain with state space {0,1,2} and transition matrix is:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

If the chain is ergodic, find the stationary distribution.

Solution:

Since the stationary distribution is verify the equation:

$$\underline{\Pi}(I-P) = \underline{0} \text{ , then:}$$

$$[\pi_1 \quad \pi_2 \quad \pi_3]\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 0\end{bmatrix}\right) = [0 \quad 0 \quad 0]$$

$$[\pi_1 \quad \pi_2 \quad \pi_3]\left(\begin{bmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1\end{bmatrix}\right) = [0 \quad 0 \quad 0]$$

$$\pi_1 - \frac{1}{2}\pi_2 - \frac{1}{2}\pi_3 = 0 \qquad \dots(1)$$

$$-\frac{1}{2}\pi_1 + \pi_2 - \frac{1}{2}\pi_3 = 0 \qquad \dots(2)$$

$$-\frac{1}{2}\pi_1 - \frac{1}{2}\pi_2 + \pi_3 = 0 \qquad \dots(3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad \dots(4)$$

Multiply equation (1), (2) and (3) by 2, we have:

$$2\pi_1 - \pi_2 - \pi_3 = 0 \qquad \dots(1)$$

$$-\pi_1 + 2\pi_2 - \pi_3 = 0 \qquad \dots(2)$$

$$-\pi_1 - \pi_2 + 2\pi_3 = 0 \qquad \dots(3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad \dots(4)$$

Subtract eq.(3) from eq.(2), we have:

$$3\pi_2 - 3\pi_3 = 0 \quad \Rightarrow \quad \pi_2 - \pi_3 = 0$$

$$\Rightarrow \quad \pi_2 = \pi_3 \qquad \dots(5)$$

Put (5) in (1), we have:

$$2\pi_1 - \pi_2 - \pi_2 = 0 \quad \Rightarrow 2\pi_1 - 2\pi_2 = 0$$

$$\Rightarrow \quad \pi_1 = \pi_2 \dots (6)$$

Then:

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

$\therefore \underline{\Pi} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, the stationary distribution.

Example (5.2):

Find the stationary distribution for this ergodic Markov chain with state space {1,2,3} and transition matrix:

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

Solution:

Since: $\underline{\Pi}(I - P) = \underline{0}$, then:

$$[\pi_1 \quad \pi_2 \quad \pi_3]\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}\right) = [0 \quad 0 \quad 0]$$

$$[\pi_1 \quad \pi_2 \quad \pi_3]\left(\begin{bmatrix} 0.7 & -0.5 & -0.2 \\ -0.6 & 1 & -0.4 \\ 0 & -0.4 & 0.4 \end{bmatrix}\right) = [0 \quad 0 \quad 0]$$

$$0.7\pi_1 - 0.6\pi_2 = 0 \qquad \ldots(1)$$

$$-0.5\pi_1 + \pi_2 - 0.4\pi_3 = 0 \qquad \ldots(2)$$

$$-0.2\pi_1 - 0.4\pi_2 + 0.4\pi_3 = 0 \qquad \ldots(3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad \ldots(4)$$

From eq.(1), we have:

$$\pi_1 = \frac{0.6}{0.7}\,\pi_2 \qquad \ldots(5)$$

put (5) in (2), we have:

$$-0.5\left(\frac{0.6}{0.7}\right)\pi_2 + \pi_2 - 0.4\,\pi_3 = 0$$

$$-0.42\,\pi_2 + \pi_2 - 0.4\,\pi_3 = 0$$

$$\pi_3 = \frac{0.58}{0.4}\pi_2 \qquad \ldots(6)$$

put (5) and (6) in (4), we have:

$$\frac{0.6}{0.7}\pi_2 + \pi_2 + \frac{0.58}{0.4}\pi_2 = 1$$

$$\therefore \pi_2 = 0.3$$

then:

$$\pi_1 = \frac{0.6}{0.7}(0.3) = 0.26$$

and:

$$\pi_3 = \frac{0.58}{0.4}(0.3) = 0.44$$

Then: $\underline{\Pi} = [0.26 \quad 0.3 \quad 0.44]$ the stationary distribution.

Remark: The relation between the stationary distribution of the states and the mean recurrent time is:

$$\mu_j = \frac{1}{\pi_j} \quad or \quad \pi_j = \frac{1}{\mu_j}$$

This is means that:

1) If $\mu_{jj} = \infty$ then the recurrent state is said to be null recurrent;

   $\pi_j = \frac{1}{\infty} = 0 \ (non - stationary)$

2) If $\mu_{jj} < \infty$ then the recurrent state is said to be positive recurrent;

   $\pi_j = \frac{1}{\mu_{jj}} > 0 \ (stationary)$