Stochastic Processes (2)

Lecture 6: Poisson process

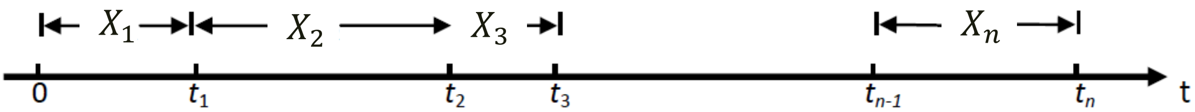
(6.1) Arrival Process:

Let t represent a time variable. Suppose an experiment begins at $t = 0$. Events of a particular kind occur randomly, the first at t_1, the second at t_2, and so on. The random variables t_i denotes the time at which the i^{th} event occurs, and the values t_i $(i = 1,2, . . .)$ are called points of occurrence.

Then the process $\{T_n, n \geq 0\}$ denotes the time from the beginning until the occurrence of the n^{th} event is called **an arrival process**.

(6.2) Interarrival Process

Let: $X_n = t_n - t_{n-1}$ and $t_0 = 0$. Then X_n denotes the time between the $(n-1)^{th}$ and the n^{th} events. The sequence of ordered r.v.'s $\{X_n , n \geq 1\}$ called an **interarrival process**. If all random variables X_n are independent and identically distributed, then $\{X_n , n \geq 1\}$ is called a **renewal process** or a **recurrent process**.



Where: $T_n = X_1 + X_2 + \cdots + X_n$

(6.3) Counting Process:

A stochastic process $\{N_t: t \geq 0\}$ is a **counting process** if N_t represents the total number of "events" that have occurred in the interval $(0, t)$. Hence counting process $\{N_t\}$ should satisfy:

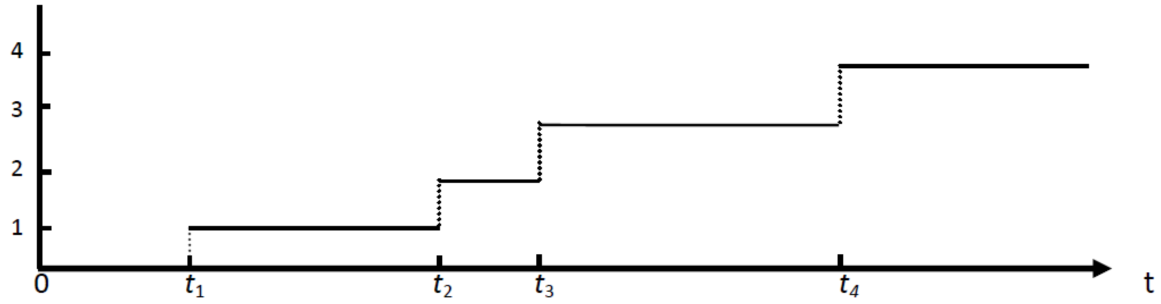
1. $N_t \geq 0$ and $X_0 = 0$
2. N_t is integer valued
3. If $s < t$, then $N_s \leq N_t$
4. For $s < t$, $N_t - N_s$ equals the number of events that have occurred on the interval $(s,\ t]$.

Examples for counting process:

- Number of persons entering a store before time t
- Number of people who were born by time t
- Number of goals a soccer player scores by time t.
- The location of users in a wireless network

A counting process N_t is said to possess **independent increments** if the numbers of events which occur in disjoint time intervals are independent.

A counting process N_t is said to possess **stationary increments** if the distribution of the number of events which occur in any interval of time depends only on the length of the time interval. i.e. the number of events in the interval $(s + h,\ t + h)$ that is, $N_{t+h} - N_{s+h}$ has the same distribution as the number of events in the interval $(s,\ t)$ that is, $N_t - N_s$ for all $s < t$ and $h > 0$.



(6.4) Poisson process:

One of the most important types of counting processes is the **Poisson process** (or Poisson counting process), which is defined as follows:

Definition:

The counting process $\{N_t : t \geq 0\}$ is said to be a Poisson process having rate (intensity) $\lambda, (\lambda > 0)$ if:

1. $N_0 = 0$.
2. The process has independent increments.
3. The number of events in any interval of length t is Poisson distributed with mean λt ; that is, for all $s, t > 0$,

$$P\{X_{t+s} - X_s = n\} = P\{X_{s,t+s} = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \quad n = 0,1, \dots$$

It follows from condition (3) that a Poisson process has stationary increments and that:

$$E[N_t] = \lambda t$$

and $$Var[N_t] = \lambda t$$

Thus, the expected number of events in the unit interval $(0,\ t)$, or any other interval of unit length, is just λ .

(6.5) The Second Definition of a Poisson Process:

The counting process $\{N_t : t \geq 0\}$ is said to be a Poisson process having rate (intensity) λ , $(\lambda > 0)$ if:

1. $N_0 = 0$.
2. The process has independent and stationary increments.
3. We have:

- $P\{N_{t+\Delta t} - N_t = 0\} = P\{N_{t,t+\Delta t} = 0\} = 1 - \lambda\,\Delta t + O(\Delta t)$
- $P\{N_{t+\Delta t} - N_t = 1\} = P\{N_{t,t+\Delta t} = 1\} = \lambda\,\Delta t + O(\Delta t)$
- $P\{N_{t+\Delta t} - N_t \geq 2\} = P\{N_{t,t+\Delta t} \geq 2\} = O(\Delta t)$

where $O(\Delta t)$ is a function of Δt which goes to zero faster than does Δt; that is: $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$

Proof of point 3:

- The probability that no event occurs in any short interval approaches unity as the duration of the interval approaches zero.

$$P\{N_{t+\Delta t} - N_t = 0\} = e^{-\lambda \Delta t}$$

$$= 1 - \lambda\,\Delta t + \frac{\lambda^2}{2!}\Delta t^2 - \cdots, \text{(Taylor Series)}$$

$$= 1 - \lambda\,\Delta t + O(\Delta t)$$

Note that if Δt is small, the terms that include second or higher powers of Δt are negligible compared to Δt.

- Now, let us look at the probability of having one arrival in an interval of length Δt.

$$P\{N_{t+\Delta t} - N_t = 1\} = \lambda \, \Delta t \, e^{-\lambda \Delta t}$$

$$= \lambda \, \Delta t \left(1 - \lambda \, \Delta t + \frac{\lambda^2}{2!} \Delta t^2 - \cdots\right), \ (\textit{Taylor Series})$$

$$= \lambda \, \Delta t + \left(-\lambda^2 \Delta t^2 + \frac{\lambda^3}{2!} \Delta t^3 - \cdots\right)$$

$$= \lambda \, \Delta t + O(\Delta t)$$

- Similarly, we can show that: $P\{N_{t+\Delta t} - N_t \geq 2\} = O(\Delta t)$

It can be shown that in the Poisson process, the intervals between successive events are independent and identically distributed exponential r.v.'s. Thus, we also identify the Poisson process as a renewal process with exponentially distributed intervals.

The counting process $\{N_t : t \geq 0\}$ is called **a Poisson process** if the inter-arrival times $X_1, X_2, \ldots$ follow the **exponential distribution.**

We conclude that the Poisson process is a discrete state, continuous time process.