Stochastic Processes (2)

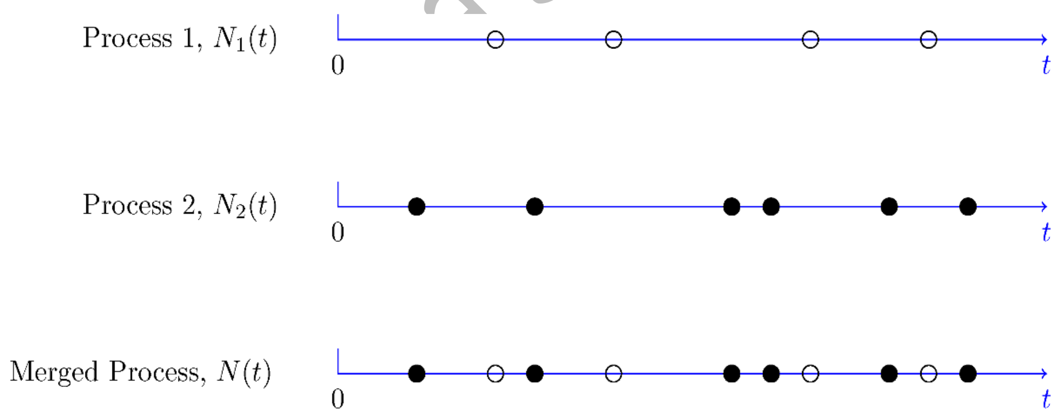
Lecture 8: Properties of Poisson Process

(8.1) Merging Independent Poisson Processes:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let us define:

$$N(t) = N_1(t) + N_2(t)$$

That is, the random process $N(t)$ is obtained by combining the arrivals in $N_1(t)$ and $N_2(t)$. In other words, $\{N(t), t \geq 0\}$ is the process consisting of all arrivals to both process 1 and process 2. We claim that $N(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.



Since $N_1(t)$ and $N_2(t)$ are independent and both have independent increments, we conclude that $N(t)$ also has independent increments.

Theorem:

Let $N_1(t),\ N_2(t),\dots,N_m(t)$ be m independent Poisson processes with rates $\lambda_1,\lambda_2,\dots,\lambda_m$ respectively. Let also:

$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t)$, for all $t \in (0,\infty]$.

Then $N(t)$ is a Poisson process with rate $(\lambda_1 + \lambda_2, + \cdots + \lambda_m)$.

Example (1):

Suppose that the cars arrive to station from three ways independent Poisson process with arrival rate (2, 4, 6) respectively. Find the probability of there is less than two cars arrive to station in 3 minutes.

Solution:

From the properties of Poisson process, we have:

$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 2 + 4 + 6 = 12 \quad , \ t = 3$

$The\ mean = \lambda t = 12(3) = 36$

$P_r\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$P_r\{N(3) < 2\} = P_r\{N(3) = 1\} + P_r\{N(3) = 0\}$

$$= \frac{e^{-36}(36)^1}{1!} + \frac{e^{-36}(36)^0}{0!} = 37e^{-36}$$

(8.2) Difference of two independent Poisson Processes:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let define $N(t) = N_1(t) - N_2(t)$. Then the random process $N(t)$ has a distribution given by:

$$P_r\{N(t) = n\} = e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_n\left(2t\sqrt{\lambda_1\lambda_2}\right)$$

When $n = 0, \mp 1, \ \mp 2, \ ...$ and:

$$I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(r+n-1)}$$

Is the modified Bessel function of order $n \geq -1$.

Proof:

$$P_r\{N(t) = n\} = \sum_{r=0}^{\infty} P_r\{N_1(t) = n + r\}.\,P_r\{N_2(t) = r\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n+r}}{(n+r)!} \ \frac{e^{-\lambda_2 t}(\lambda_2 t)^{r}}{r!}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} \sum_{r=0}^{\infty} \frac{\left(t\sqrt{\lambda_1\lambda_2}\right)^{2r+n}}{r!(r+n)!}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_n\left(2t\sqrt{\lambda_1\lambda_2}\right)$$

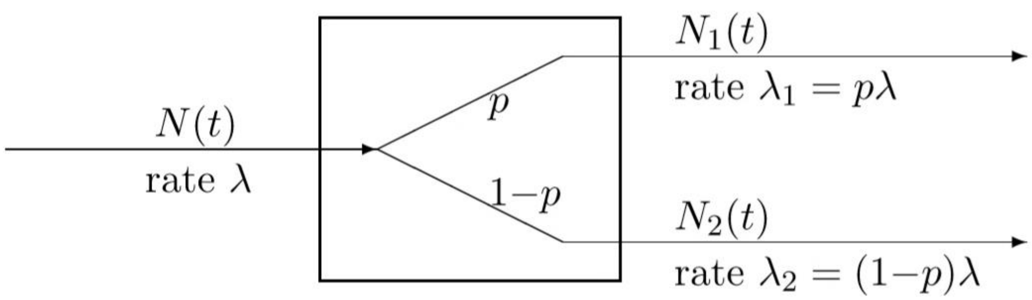
Where: $I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(r+n-1)}$

Thus, the difference of two independent Poisson process is not Poisson process.

(8.3) Splitting of Poisson Processes:

Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way (Figure blew). For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process $N_1(t)$, otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,

1. $N_1(t)$ is a Poisson process with rate λp.
2. $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$.
3. $N_1(t)$ and $N_2(t)$ are independent.



Each arrival is independently sent to process 1 with probability p and to process 2 otherwise.

Example (2):

Suppose that the sale of computer in a shop is follows Poisson process with rate $\lambda = 7$ computer per day, and if the number of type laptop computers sold 3 of 10 computers per day, find the probability that:

1) There are four laptops sold in two days.

2) Sale three computers in two days not of type laptop.

3) Sale five computers per day.

Solution:

1) Let $N_1(t)$ sale laptop in shop with Poisson process with probability:

$$p = \frac{3}{10} = 0.3$$

then the rate of sale laptop is:

$\lambda_1 = \lambda p = 7(0.3) = 2.1$, then:

$$P_r\{N_1(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$P_r\{N_1(2) = 4\} = \frac{e^{-(2.1)(2)}((2.1)(2))^4}{4!} = 12.97\, e^{-4.2}$$

2) Let $N_2(t)$ sale computers of type not laptop in shop with Poisson process with probability:

$1 - p = 1 - 0.3 = 0.7$

Then the rate sale computers of type not laptop is:

$\lambda_2 = \lambda(1 - p) = 7(0.7) = 4.9$, then:

$$P_r\{N_2(t) = n\} = P_r\{N_2(2) = 3\}$$

$$= \frac{e^{-(4.9)(2)}((4.9)(2))^3}{3!} = 156.8\, e^{-9.8}$$

3) Sale five computers from shop, then: $n = 5\,,\ t = 1\,,\ \lambda = 7$

$$P_r\{N(t) = n\} = P_r\{N(1) = 5\} = \frac{e^{-7(1)}(7(1))^5}{5!} = 140.05\, e^{-7}$$