Stochastic Processes (2)

Lecture 9: Solved Problems

Problem (1):

Suppose that patients arriving at a hospital through three cities follow a Poisson process with arrival rates (1, 3, 5) per minute respectively. Find the probability of more than or equal two patients arriving at the hospital in 3 minutes.

Solution

From the properties of the Poisson process, we have:

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 3 + 5 = 9 \quad , \quad t = 3$$

$$The\ mean = \lambda t = 9(3) = 27$$

$$P_r\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$P_r\{N(3) \geq 2\} = 1 - P_r\{N(3) < 2\}$$

$$= 1 - [P_r\{N(3) = 1\} + P_r\{N(3) = 0\}]$$

$$= 1 - \left[\frac{e^{-27}(27)^1}{1!} + \frac{e^{-27}(27)^0}{0!}\right]$$

$$= 1 - 28e^{-27}$$

Problem (2):

A call center receives an average of 10 calls per hour. Let $N(t)$ be the number of calls that have arrived up to time t. Determine the following probabilities and conditional probabilities:

1. $P(N(1) = 3)$, i.e., the probability that exactly 3 calls arrive in the first hour.
2. $P(N(3) > 2)$, i.e., the probability that more than 2 calls arrive in the first three hours.
3. $P(N(2) = 3 \mid N(1) = 1)$, i.e., the probability that exactly 3 calls arrive in the second hour given that exactly one call arrived in the first hour.

Solution

1. $P(N(1) = 3) = \frac{e^{-(10)(1)}\left((10)(1)\right)^3}{3!} = 166.67\, e^{-10}$

2. $P(N(3) > 2) = 1 - P(N(3) \le 2)$

$$= 1 - [P(N(3) = 0) + P(N(3) = 1) + P(N(3) = 2)]$$

$$= 1 - \left[\frac{e^{-(10)(3)}\left((10)(3)\right)^0}{0!} + \frac{e^{-(10)(3)}\left((10)(3)\right)^1}{1!} + \frac{e^{-(10)(3)}\left((10)(3)\right)^2}{2!}\right]$$

$$= 1 - [e^{-30} + 30\, e^{-30} + 450\, e^{-30}]$$

$$= 1 - 481\, e^{-30}$$

3. $P(N(2) = 3|N(1) = 1) = \frac{P(N(2)=3\ ,N(1)=1)}{P(N(1)=1)}$

$$= \frac{P\{N(2-1)=(3-1)\}.P\{N(1)=1\}}{P(N(1)=1)}$$

$$= P\{N(1) = 2\} = \frac{e^{-10(1)}(10(1))^2}{2!} = 50\ e^{-10}$$

Problem (3):

Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $N(t)$ be the number of defects, either major or minor, in the cable up to length t. Discuss that $N(t)$ must be a Poisson process of rate $\alpha + \beta$.

Solution

Let $N_1(t)$ and $N_2(t)$ be the minor and major defects independent Poisson processes with rates α and β respectively. Then the number of defects (major or minor) in the cable up to length t is:

$$N(t) = N_1(t) + N_2(t)$$

Since $N_i(t), i = 1,2$, distributed as Poisson distribution, then the p.g.f. of $N_i(t)$ are:

$$P_{N_1(t)}(S) = \sum_{k=0}^{\infty} p_k S^k$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\alpha t}\,(\alpha t)^k}{k!} S^k = e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t S)^k}{k!} = e^{-\alpha t}\ e^{\alpha t S}$$

$$= e^{\alpha t(S-1)}$$

And also $P_{N_2(t)}(S) = e^{\beta t(S-1)}$

Then the p.g.f. of $N(t)$ is:

$$P_{N(t)}(S) = P_{N_1(t)}(S) \, . P_{N_2(t)}(S) \quad \text{, [from theorem (1)]}$$

$$= e^{\alpha t(S-1)} \, . e^{\beta t(S-1)}$$

$$= e^{(\alpha+\beta)t(S-1)}$$

Then the distribution of $N(t)$ is a Poisson distribution, then $N(t)$ is a Poisson Process with rate $(\alpha + \beta)$.

Problem (4):

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

a. Find the probability that $N(1) = 2$ and $N(2) = 5$.
b. Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Solution

$N(t)$ is a Poisson process with rate $\lambda = 1 + 2 = 3$.

a. We have

$$P(N(1) = 2, N(2) = 5) = P\Big(\underline{two} \text{ arrivals in } (0,1] \text{ and } \underline{three} \text{ arrivals in } (1,2]\Big)$$
$$= \left[\frac{e^{-3}3^2}{2!}\right] \cdot \left[\frac{e^{-3}3^3}{3!}\right]$$
$$\approx .05$$

b.

$$P(N_1(1) = 1 | N(1) = 2) = \frac{P\big(N_1(1) = 1, N(1) = 2\big)}{P(N(1) = 2)}$$
$$= \frac{P\big(N_1(1) = 1, N_2(1) = 1\big)}{P(N(1) = 2)}$$
$$= \frac{P\big(N_1(1) = 1\big) \cdot P\big(N_2(1) = 1\big)}{P(N(1) = 2)}$$
$$= \left[e^{-1} \cdot 2e^{-2}\right] / \left[\frac{e^{-3}3^2}{2!}\right]$$
$$= \frac{4}{9}.$$