Power Series

متسلسلات القوى

It is aseries whose base is x and which has the exponent $(n+1)$.

i.e.:-

$$V_n(x) = 1 + a_0 x + a_1 x^2 + a_2 x^3 + \dots + a_n x^{n+1}$$

is polynomial

It is types as follows:-

① Tayler Series:-

The general Series:-

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots + \frac{f^n(a)(x-a)^n}{n!}$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

Ex Find taylor series of the function

1. $f(x) = (1+x)^3$: at $a=1$?

solu

$f(a) = (1+1)^3 = 8$

$f'(a) = 3(1+x)^2 \Rightarrow f'(1) = 3(2)^2 = 12$

$f''(a) = 6(1+x) \Rightarrow f''(1) = 6(2) = 12$

$f'''(a) = 6 \Rightarrow f''(1) = 6$

$f''''(a) = 0$

$$\therefore f(x) = 8 + \frac{12(x-1)}{1!} + \frac{12(x-1)^2}{2!} + \frac{6(x-1)^3}{3!} + 0$$

$$f(x) = 8 + 12(x-1) + 6(x-1)^2 + (x-1)^3$$

2. $f(x) = x^4 - 3x^2 + 1 \quad ; \quad a = 1$

Solu

$n=0 \Rightarrow f(x) = x^4 - 3x^2 + 1 \Rightarrow f(1) = 1 - 3 + 1 = -1$

$n=1 \Rightarrow f'(x) = 4x^3 - 6x \Rightarrow f'(1) = 4 - 6 = -2$

$n=2 \Rightarrow f''(x) = 12x^2 - 6 \Rightarrow f''(1) = 12 - 6 = 6$

$n=3 \Rightarrow f'''(x) = 24x \Rightarrow f^3(1) = 24$

$n=4 \Rightarrow f^4(x) = 24 \Rightarrow f^4(1) = 24$

$n=5 \Rightarrow f^5(x) = 0 \Rightarrow f^5(1) = 0$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

$$f(x) = \frac{(-1)(x-1)^0}{0!} + \frac{(-2)(x-1)^1}{1!} + \frac{6(x-1)^2}{2!} + \frac{24(x-1)^3}{3!} +$$

$$\frac{24(x-1)^4}{4!} + \frac{0(x-1)^5}{5!} + \dots + \frac{0(x-1)^n}{n!}$$

$$= (-1) + (-2)(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4 + \dots + 0$$

$$\therefore x^4 - 3x^2 + 1 = -1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4$$

3. $f(x) = e^{2x} \quad ; \quad a = 3$

Solu $\quad f(x) = e^{2x} \quad ; \quad f(3) = e^{6}$

$f'(x) = 2e^{2x} \quad ; \quad f'(3) = 2e^{2(3)} = 2e^{6}$

$f''(x) = 4e^{2x} \quad ; \quad f''(3) = 4e^{6}$

$f^{3}(x) = 8e^{2x} \quad ; \quad f^{3}(3) = 8e^{6}$

$f^{4}(x) = 16e^{2x} \quad ; \quad f^{4}(3) = 16e^{6}$

$\vdots$

$f^{n}(x) = 2^{n}e^{2x} \quad ; \quad f^{n}(3) = 2^{n}e^{6}$

$$\therefore f(x) = \frac{e^{6}(x-3)^{0}}{0!} + \frac{2e^{6}(x-3)^{1}}{1!} + \frac{4e^{6}(x-3)^{2}}{2!} + \dots + \frac{f^{n}(a)(x-3)^{n}}{n!}$$

$$= \frac{e^{6}(x-3)^{0}}{0!} + \frac{2e^{6}(x-3)^{1}}{1!} + \frac{4e^{6}(x-3)^{2}}{2!} + \dots + \frac{2^{n}e^{6}(x-3)^{n}}{n!}$$

$$= e^{6} + 2e^{6}(x-3) + 2e^{6}(x-3)^{2} + \dots + \frac{2^{n}e^{6}(x-3)^{n}}{n!}$$

$$\therefore e^{2x} = e^{6} + 2e^{6}(x-3) + 2e^{6}(x-3)^{2} + \dots + \frac{2^{n}e^{6}(x-3)^{n}}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^{n}e^{6}(x-3)^{n}}{n!}$$

H.W. Find taylor series of the function:

$$f(x) = \ln x \quad \text{at} \quad a = 2$$

Solu

$n=0 \Rightarrow f(x) = \ln x \quad ; \quad f(2) = \ln 2$

$n=1 \Rightarrow f'(x) = \dfrac{1}{x} \quad ; \quad f'(2) = \dfrac{1}{2}$

$n=2 \Rightarrow f''(x) = \dfrac{-1}{x^2} \quad ; \quad f''(2) = \dfrac{-1}{4}$

$n=3 \Rightarrow f^3(x) = \dfrac{2}{x^3} \quad ; \quad f^3(2) = \dfrac{2}{8} = \dfrac{1}{4}$

$n=4 \Rightarrow f^4(x) = \dfrac{-6}{x^4} \quad ; \quad f^4(2) = \dfrac{-6}{16} = \dfrac{-3}{8}$

$n=n \Rightarrow f^n(x) = \dfrac{(-1)^{n+1}(n-1)!}{x^n} \quad ; \quad f^n(2) = \dfrac{(-1)^{n+1}(n-1)!}{2^n}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

$$= \frac{f^0(a)(x-2)^0}{0!} + \frac{f'(a)(x-2)^1}{1!} + \frac{f''(a)(x-2)^2}{2!} + \dots + \frac{f^n(a)(x-2)^n}{n!}$$

$$\ln x = \frac{\ln(2)(x-2)^0}{0!} + \frac{\frac{1}{2}(x-2)^1}{1!} - \frac{\frac{1}{4}(x-2)^2}{2!} + \dots + \frac{(-1)^{n+1}(n-1)!}{2^n}$$

$$\therefore \ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)(x-2)^n}{2^n\, n!}$$