Integrations

Integration is the process of finding the function whose derivative is known,
i.e.: It is the reverse process of finding the derivative.

There are two types of integration:-

Type one :- definite integration

It is calculated as follows:-

$$\int_a^b f(x)\,dx$$

Type two:- Indefinite integration

It is calculated as follows:-

$$\int f(x)\,dx$$

The basic formula for integration is:-

Add one to the exponent and divide by the new exponent, As shown in the following:-

$$\int u^m\,du = \frac{u^{m+1}}{m+1} + c$$

The properties of integration are:-

1. If $f(k) = k$, when k is constant, Then

$$\int f(x)\,dx = k\int dx = kx + c$$

2. If $f(x)$ and $g(x)$ any functions, then:-

$$\int (f(x) \mp g(x))\,dx = \int f(x)\,dx \mp \int g(x)\,dx$$

3. $$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Ex

1. $$\int x\,dx = \frac{x^2}{2} + c$$

2. $$\int x^2\,dx = \frac{x^3}{3} + c$$

3. $$\int (x^3 + 4x)\,dx = \int x^3\,dx + \int 4x\,dx$$

$$= \frac{x^4}{4} + \frac{4x^2}{2} + c$$

4. $$\int_0^1 (3x^2 + 2)\,dx = \int_0^1 3x^2\,dx + \int_0^1 2\,dx$$

$$= \frac{3x^3}{3}\Big|_0^1 + 2x\Big|_0^1 = (1-0) + (2-0)$$

$$= 1 + 2 = 3$$

5. $\int \cos x - (\csc^2 x + 1)\, dx$

$= \sin x + \cot x + x + c$

6. $\int \frac{\cos x}{\sin^2 x}\, dx = \int \left(\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}\right) dx$

$= \int \csc x\, dx \cdot \int \cot x\, dx$

$= -\csc x + c$

or $\int \cos x \sin^{-2} x\, dx = -\sin^{-1} x$

$= \frac{-1}{\sin x} = -\csc x + c$

7. $\int \left(\frac{x^2 - 2x^4}{x^4}\right) dx = \int \frac{x^2}{x^4}\, dx - \int \frac{2x^4}{x^4}\, dx$

$= \frac{-1}{x} - 2x + c$

8. $\int \frac{1}{1+e^x}\, dx$ $\qquad \mp e^x$

$= \int \frac{1 + e^x - e^x}{1+e^x}\, dx$

$= \int \left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right) dx$

$$= \int \left(1 - \frac{e^x}{1+e^x}\right) dx$$

$$= x - \ln|1+e^x| + c$$

9. $\int 7 \cos x \, dx = 7 \sin x + c$

10. $\int (5x^4 + 2x)\, dx = \int 5x^4 dx + \int 2x \, dx$

$$= \frac{5x^5}{5} + \frac{2x^2}{2} = x^5 + x^2 + c$$

H.w.

1. $\int (3x^2 - 9x + 1)^{1/2} (2x - 3)$

2. $\int (3x+1)^{-\frac{2}{3}} dx$

3. $\int (ax^2 + bx^3 + c) dx$

4. $\int \sqrt{2x+1}\, dx$

5. $\int \frac{1}{x} dx = \ln|x| + c$

6. $\int \frac{3}{3x+1} dx = \ln|3x+1| + c$

Methods of Integrations:-

1) Integrations by Substitution بالاستبدال

This technique can often be used to transform complicated integration problems into simple ones.

This chain rule implies that as:-

$$\frac{\partial F g(x)}{\partial x} = F' g(x) . g'(x)$$

which we can write in integral forms:-

$$\int F'(g(x) . g'(x)) dx = F'(g(x)) + c$$

For our purposes it will be useful to let $u = g(x)$ and write:

$$\frac{du}{dx} = g'(x) \quad \text{in the differential form}$$

$$du = g'(x) dx$$

So we can be expressed as:

$$\int f(u) du = F(u) + c$$

For example:-

1. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$\therefore 2du = \frac{dx}{\sqrt{x}} \Rightarrow dx = 2\sqrt{x}\, du$

$= 2u\, du$

$\therefore \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = \int \frac{\cos u}{u} 2u\, du$

$= \int \cos u \cdot 2\, du$

$= 2 \sin u + c = 2 \sin\sqrt{x} + c$

2. $\int (\csc(\sin x))^2 \cdot \cos x\, dx$

let $u = \sin x \Rightarrow du = \cos x\, dx$

$\therefore \int (\csc u)^2 \cdot du$

$= -\cot u + c = -\cot(\sin x) + c$

3. $\int \frac{y}{\sqrt{y+1}} dy$

let $u = \sqrt{y+1} \Rightarrow u^2 = y+1 \Rightarrow y = u^2 - 1$

$\therefore dy = 2u\, du$

$\therefore \int \frac{u^2-1}{u} 2u\, du$

$$= \int (2u^2 - 2)\,du$$

$$= \frac{2u^3}{3} - 2u + c = \frac{2}{3}(\sqrt{y+1})^3 - 2\sqrt{y+1} + c$$

5. $\int x\sqrt{x-1}\,dx$

let $u = \sqrt{x-1} \Rightarrow u^2 = x - 1 \Rightarrow x = u^2 + 1$

$dx = 2u\,du$

$$\therefore \int x\sqrt{x-1}\,dx = \int u^2+1\,\sqrt{u^2+1-1}\;2u\,du$$

$$= \int (u^2+1)u \cdot 2u\,du$$

$$= \int (u^2+1) \cdot 2u^2\,du$$

$$= \int (2u^4 + 2u^2)\,du$$

$$= \frac{2}{5}u^5 + \frac{2}{3}u^3 + c$$

$$= \frac{2}{5}(\sqrt{x-1})^5 + \frac{2}{3}(\sqrt{x-1})^3 + c$$

H.W. Evalute the following :-

1) $\displaystyle\int \frac{\sin 2\theta}{5+\cos 2\theta}\, d\theta = \frac{1}{2} \ln|\cos(2\theta+5)+c$

2) $\displaystyle\int x^2\sqrt{1+x}\, dx = \frac{2}{7}(\sqrt{1+x})^7 - \frac{4}{5}(\sqrt{1+x})^5 + \frac{2}{3}(\sqrt{1+x})^3 + c$

3) $\displaystyle\int \frac{1}{x^2} \sin\left(\frac{5}{x}\right) dx = \int \frac{\sin\left(\frac{5}{x}\right)}{x^2}\, dx$

$\displaystyle = \frac{1}{5} \cos \frac{5}{x} + c$

4) $\displaystyle\int \cos(\sin x) \cos x\, dx$

let $y = \sin x \Rightarrow dy = \cos x\, dx$

$\displaystyle\therefore \int \cos y\, dy = \sin y + c = \sin(\sin x) + c$

5) $\displaystyle\int x \cos x^2\, dx$

let $u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx$

$\displaystyle\therefore \int x \cos x^2\, dx = \int \frac{1}{2} \cos y\, dy = \frac{1}{2} \sin y + c$

$\displaystyle = \frac{1}{2} \sin x^2 + c$

6) $\displaystyle\int x^2 \sqrt{x^3+5}\, dx$; let $u = x^3+5 \Rightarrow du = 3x^2\, dx$

2) Integration by Parts

Let u and v any two function of x, then:

$$\frac{\partial}{\partial x}(u \cdot v) = u\frac{dv}{dx} + v\frac{\partial u}{dx}$$

By integrating both sides we get:-

$$\int \frac{\partial}{\partial x}(u \cdot v)\,dx = \int u\frac{dv}{\partial x}\,dx + \int v\frac{du}{dx}\,dx$$

$$u \cdot v = \int u\,dv + \int v\,du$$

$$\therefore \int u\,dv = u \cdot v - \int v\,du$$

Ex

Evalute the following integrals:-

1. $\int x \sin x \, dx$

let $u = x \Rightarrow du = dx$

$dv = \sin x \, dx \Rightarrow v = -\cos x$

$$\therefore \int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

2. $\int \tan^{-1} dx$

let $u = \tan^{-1}x \Rightarrow du = \frac{1}{1+x^2}dx$

$dv = dx \Rightarrow v = x$

$\therefore \int \tan^{-1} dx = x\tan^{-1}x - \int x \frac{1}{1+x^2} dx$

$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + c$

$= x\tan^{-1}x + \ln\frac{1}{\sqrt{1+x^2}} + c$

3. $\int e^x \sin x\, dx$

let $u = e^x \Rightarrow du = e^x dx$

$dv = \sin x\, dx \Rightarrow v = -\cos x$

$\therefore \int e^x \sin x\, dx = -e^x\cos x + \int e^x \cos x\, dx$

let $u_1 = e^x \Rightarrow du = e^x dx$

$dv_1 = \cos x\, dx \Rightarrow v = \sin x$

$\therefore \int e^x \sin x\, dx = -e^x \cos x + \left[e^x \sin x - \int e^x \sin x\, dx\right]$

let $u_2 = \sin x \Rightarrow du_2 = \cos x\, dx$

$dv_2 = e^x\, dx \Rightarrow v_2 = e^x$

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \left[e^x \sin x - \int e^x \cos x\, dx\right]$$

Since the function take the same pattern for three stages, we stop and call the function a peridic function.

$I = -e^x \cos x + e^x \sin x - e^x \sin x - I$

$2I = -e^x \cos x$

$I = -\frac{1}{2} e^x \cos x + c$

4. $\int x^2 \sqrt{x-1}\, dx$

let $u = x^2 \Rightarrow du = 2x\, dx$

$dv = (x-1)^{1/2} dx \Rightarrow v = \frac{2}{3}(x-1)^{\frac{3}{2}}$

$$\therefore \int x^2 \sqrt{x-1}\, dx = \frac{2}{3} x^2 (x-1)^{\frac{3}{2}} - \int \frac{4}{3} x (x-1)^{\frac{3}{2}} dx$$

let $u_1 = x \Rightarrow du = dx$

$dv_1 = (x-1)^{\frac{3}{2}} dx \Rightarrow dv = \frac{2}{5}(x-1)^{\frac{5}{2}}$

$$\therefore \int x^2\sqrt{x-1}\,dx = \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\left[\frac{2}{5}x(x-1)^{\frac{5}{2}} - \int \frac{2}{5}(x-1)^{\frac{5}{2}}\,dx\right.$$

$$= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8}{15}x(x-1)^{\frac{5}{2}} - \frac{16}{105}(x-1)^{\frac{7}{2}} + c$$

5. $\int x\ln x\,dx$

let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$\therefore \int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$$

6. $\int x^3\ln x\,dx$

let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

$dv = x^3\,dx \Rightarrow v = \frac{x^4}{4}$

$$\therefore \int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + c$$