5. Ex $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx$

To integrate aquantity in this form we find the Common multiple as follows:-

$\sqrt[2]{x}\ ,\ \sqrt[4]{x}$

المضاعف المشترك هو $\sqrt[4]{x}$

let $y=\sqrt[4]{x} \Rightarrow y=x^{\frac{1}{4}}$

$y^4=x \Rightarrow dx=4y^3\,dy$

$$\therefore \int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx=\int \frac{y^2}{1+y}\,4y^3\,dy$$

$$=\int \frac{4y^5}{1+y}\,dy$$

$$=4\left(\int (y^4-y^3+y^2-y+1)\,dy-\int \frac{1}{1+y}\,dy\right)$$

$$=\frac{4}{5}y^5-\frac{4}{4}y^4+\frac{4}{3}y^3-\frac{4}{2}y^2+4y-4\ln|1+y|+c$$

$$=\frac{4}{5}x^{\frac{5}{4}}-x+\frac{4}{3}x^{\frac{3}{4}}+4x^{\frac{1}{4}}+4\ln|1+x^{\frac{1}{4}}|+c$$

$$
\begin{array}{r|l}
 & y^4-y^3+y^2-y+1 \\
\hline
1+y & y^5 \\
 & \mp y^5 \mp y^4 \\
\hline
 & -y^4 \\
 & \pm y^4 \pm y^3 \\
\hline
 & y^3 \\
 & \mp y^3 \mp y^2 \\
\hline
 & -y^2 \\
 & \pm y^2 \pm y \\
\hline
 & y \\
 & \mp y \mp 1 \\
\hline
 & -1
\end{array}
$$

6. $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

$\sqrt{x}\ ,\ \sqrt[3]{x}$

$\sqrt[6]{x}$

let $y=\sqrt[6]{x} \Rightarrow y^6 = x \Rightarrow dx = 6y^5\,dy$

$$\therefore \int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \int \frac{y^3}{1+y^2}\,6y^5\,dy$$

$$= 6\int \frac{y^8}{1+y^2}$$

$$= 6\left[\int (y^6-y^4+y^2-1)\,dy + \int \frac{1}{1+y^2}\,dy\right]$$

$$= \frac{6}{7}y^7 - \frac{6}{5}y^5 + \frac{6}{3}y^3 - 6y + 6\tan^{-1}y + c$$

$$= \frac{6}{7}(x)^{\frac{7}{6}} - \frac{6}{5}(x)^{\frac{5}{6}} + \frac{6}{3}(x)^{\frac{1}{2}} - 6(x)^{\frac{1}{6}} + 6\tan^{-1}\sqrt[6]{x}$$

$$= \frac{6}{7}(x)^{\frac{7}{6}} - \frac{6}{5}(x)^{\frac{5}{6}} - 2\sqrt{x} - 6(x)^{\frac{1}{6}} + 6\tan^{-1}\sqrt[6]{x} + c$$

$$\begin{array}{r|l}
 & y^6 - y^4 + y^2 - 1 \\
1+y^2 & y^8 \\
 & \mp y^8 \mp y^6 \\
\hline
 & -y^6 \\
 & \pm y^6 \pm y^4 \\
\hline
 & y^4 \\
 & \mp y^4 \mp y^2 \\
\hline
 & -y^2 \\
 & \pm y^2 \pm 1 \\
\hline
 & 1
\end{array}$$

H.w.

7 $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}+x}$

let $y=\sqrt{x} \Rightarrow y^2 = x \Rightarrow dx = 2y\,dy$

$$\therefore \int \frac{x\,dx}{1+\sqrt{x}+x} = \int \frac{y^2 \cdot 2y\,dy}{1+y+y^2}$$

$$= 2\int \frac{y^3}{1+y+y^2}\,dy$$

$$\begin{array}{r} y-1 \\ y^2+y+1 \,\overline{)\, y^3 \qquad\quad} \\ \mp y^3 \mp y^2 \mp y \\ \hline -y^2 - y \\ \pm y^2 \pm y \pm 1 \\ \hline 1 \end{array}$$

$$= 2\left[\int (y-1)\,dy + \int \frac{1}{y^2+y+1}\,dy \right]$$

$$= \frac{2}{2}y^2 - 2y + 2\int \frac{dy}{y^2+y+1+\frac{1}{4}-\frac{1}{4}}$$

$$= y^2 - 2y + 2\int \frac{dy}{(y+\frac{1}{2})^2+\frac{3}{4}}$$

$$= y^2 - 2y + 2\int \frac{dy}{\frac{3}{4}\left(1+\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2\right)} \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}$$

$$= y^2 - 2y + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= x - 2\sqrt{x} + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{x}+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

4. Integration using Completing the Square method

This method is used when there is a quadratic equation that cannot be solve by fractional division, as follows:-

Ex

1. $$\int \frac{1}{x^2+2x+2}\,dx$$

we take the square of half the coefficient of x, that is:- $\mp\left(\frac{1}{2}x\right)^2$

$$\therefore = \int \frac{dx}{x^2+2x+2+1-1}$$

$$= \int \frac{dx}{x^2+2x+1+(2-1)}$$

$$= \int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1)+c$$

or

let $u = x+1 \Rightarrow du = dx$

$$= \int \frac{du}{u^2+1} = \tan^{-1}u + c = \tan^{-1}(x+1)+c$$

2. $\int \frac{dx}{x-2\sqrt{(x^2-4x+3}}$ ∓ 4

Solu

$$= \int \frac{dx}{x-2\sqrt{x^2-4x+3+4-4}}$$

$$= \int \frac{dx}{x-2\sqrt{(x-2)^2+(3-4)}}$$

$$= \int \frac{dx}{x-2\sqrt{(x-2)^2-1}} = \sec^{-1}(x-2)+c$$

or let $u = x-2 \Rightarrow du = dx$

$$= \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}u + c = \sec^{-1}(x-2)+c$$

3. $\int \frac{x-4}{x^2-6x+10}\,dx$ ∓ 9

Solu $= \int \frac{x-4}{x^2-6x+10+9-9}\,dx = \int \frac{x-4}{x^2-6x+9+(10-9)}\,dx$

$$= \int \frac{x-4}{(x-3)^2+1}\,dx$$

let $u = x-3 \Rightarrow du = dx \Rightarrow x = u+3$

$$= \int \frac{u+3-4}{u^2+1}\,du = \int \frac{u-1}{u^2+1}\,du$$

$$= \int \frac{u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\ln|u^2+1| - \tan^{-1}u + c$$

$$= \frac{1}{2}\ln|(x-3)^2+1| - \tan^{-1}(x-3) + c$$

3. $$\int \frac{dx}{20+8x-x^2} \qquad \mp 16$$

Sol:

$$= \int \frac{dx}{\sqrt{-(x^2-8x-20)}} = \int \frac{dx}{\sqrt{-(x^2-8x-20+16-16}}$$

$$= \int \frac{dx}{\sqrt{-(x^2-8x+6)+36}} = \int \frac{dx}{\sqrt{-(x-4)^2+36}}$$

$$= \int \frac{dx}{\sqrt{36-(x-4)^2}} = \int \frac{dx}{6\sqrt{1-\left(\frac{x-4}{6}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x-4}{6}\right) + c$$